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FLUID MECHANICS

ANALYTICAL STUDY OF PARALLEL PLATE FLOW OF A NEWTONIAN FLUID WITH SPATIAL VARIATIONS IN VISCOSITY

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ABSTRACT: One dimensional Newtonian fluid flow between fixed parallel plates is considered. The viscosity is not constant but changes with the spatial variable of height. Two different classes of problems are treated: 1) Continuous changes in the viscosity, 2) Discontinuous changes in the viscosity. In the first class, general integral form solutions are given first and then, for the four different viscosity functions, the velocities and the discharges are calculated. For the second class of discontinuous changes, two problems are solved: 1) Two different immiscible fluids one laying on top of the other, 2) An intermediate high viscosity fluid lubricated by a low viscosity thin layer fluid at the boundaries. Velocity profiles and discharges are calculated for both cases. It is found that spatial viscosity variations have substantial effects on the velocity profiles and discharges and cannot be ignored for a precise treatment.

KEY WORDS: Newtonian Fluids, Parallel Plate Flow, Variable Viscosity, Analytical Solutions.

1 INTRODUCTION

The motion of a viscous fluid is governed by Navier-Stokes equations. The equations possess exact analytical solutions only for some very restricted flow conditions. One of them is the one-dimensional parallel plate flow which is treated in standard textbooks on fluid mechanics. Due to its simple flow geometry, the problem was treated extensively for Newtonian and non-Newtonian fluids [1–10]. Numerical solutions as well as approximate analytical solutions (Perturbation Method, Perturbation Iteration Method, Adomian Decomposition Method) were presented for the problems considered in the mentioned references.

In this work, flow of a Newtonian fluid in between fixed parallel plates is considered. Viscosity variations with the spatial coordinate is taken into account. For

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fluids with spatial dependent variation of viscosities for different flow conditions, see [11–16]. A power-law variation with respect to the radial coordinate was taken in [11, 15]. Blood viscosity was also modeled as a power-law function of the radial coordinate [14]. A complex relationship involving hyperbolic sine functions were considered for spatial variations [12]. Quadratic dependence of viscosity on the spatial coordinates of height and radial distance was separately investigated in [13]. An exponential type variation is assumed in [16]. For inhomogenous fluids, colloids, suspensions and biological fluids, spatial variations in viscosities indeed occur. First, the parallel plate flow with continuous variation of the viscosity with height is solved in a general form with the velocity profiles and discharges given in integral forms. The integrals are then evaluated for four different spatial variation functions. Instead of a continuous change, a discontinuous change is also considered for two different problems. In the first one, two different viscosity fluids lay on each other. In the second problem, the high viscosity intermediate fluid is lubricated with a low viscosity fluid adjacent to the boundaries. Velocity profiles and discharges are calculated for the problems. It is found that the spatial variations in viscosities may have non-negligible effects on the velocity profiles and discharges. To the best of the authors' knowledge, the solutions presented in this work were not reported before.

2 Velocity and Discharge

The geometry of the problem is given in Fig. 1.

For a Newtonian fluid flowing in between fixed parallel plates with spatially varying viscosity, the Navier-Stokes equations reduce to

$$\frac{d}{dy^*}\left(\mu^*(y^*)\frac{du^*}{dy^*}\right) = P_G^*\,, \tag{1}$$

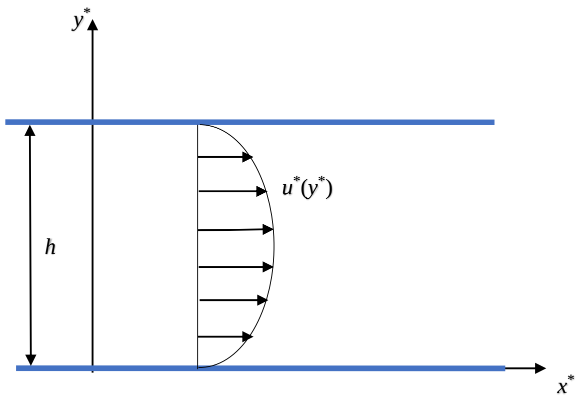

Fig. 1: Geometry of the problem.

where P_G^* is the constant pressure gradient in the x^* direction, $u^*(y^*)$ is the velocity in the x^* direction, y^* is the vertical coordinate and $\mu^*(y^*)$ is the height dependent viscosity. The dimensional quantities (denoted by asterisk) are made nondimensional by defining the dimensionless quantities

$$y = \frac{y^*}{h}, \quad u = \frac{u^*}{u_0}, \quad \mu = \frac{\mu^*}{\mu_0}, \quad \Lambda = \frac{P_G^*}{P_G}, \tag{2}$$

where h is the distance between the plates, u_0 and μ_0 are some reference velocity and viscosity. Note that, with reference to Figure 1, the dimensional pressure gradient P_G^* and the dimensionless pressure gradient Λ should be selected negative to maintain the positive velocity components. Substituting (2) into (1) and selecting

$$P_G = \frac{\mu_0 u_0}{h^2}, \tag{3}$$

the dimensionless equation of motion turns out to be

$$\frac{d}{dy}\left(\mu(y)\frac{du}{dy}\right) = \Lambda\,. \tag{4}$$

The boundary conditions for the problem are

$$u\,(0) = 0, \quad u\,(1) = 0\,. \tag{5}$$

Integrating twice the equation, the velocity profiles are

$$u\,(y) = \int_0^y \frac{\Lambda y + c}{\mu(y)} dy\,, \tag{6}$$

where

$$c = -\Lambda \int_0^1 \frac{y dy}{\mu(y)} \Big/ \int_0^1 \frac{dy}{\mu(y)}\,. \tag{7}$$

The dimensionless discharge per unit width is

$$Q = \int_0^1 \int_0^y \frac{\Lambda y + c}{\mu(y)} dy dy \tag{8}$$

Equations (6)-(8) are the general solutions for arbitrary viscosity functions. Specific cases will be treated in the next section.

3 CONTINUOUS VISCOSITY VARIATIONS

Four special cases of viscosity functions are treated in this section. The functions are chosen as simple as they can be to outline the fundamental effects of spatial variation of viscosity on the flow. Since experimental results are lacking in the literature, the comparisons of the theoretical solutions cannot be done. Experiments in future however may lead to more complex empirical relationships for spatial variations of viscosities. Section 2 outlines the general algorithm for arbitrary spatial variation viscosities and velocities and discharges can easily be calculated from the formulas for complex spatial variations.

3.1 $\mu = 1$ CASE

This is the constant viscosity case in dimensionless form. The velocity profile for the problem is

$$u\,(y) = \frac{\Lambda}{2} y(y-1)\,, \tag{9}$$

which is the well-known parabolic solution. The discharge is

$$Q = -\frac{\Lambda}{12}\,. \tag{10}$$

3.2 $\mu = \dfrac{1}{1+\alpha y}$ CASE

Depending on the density reduction with height, the viscosity may also drop. This effect is more prominent for gases compared to liquids. In this case, it can be assumed that the viscosity of the fluid varies with an inversely proportional relationship with the height coordinate. Substituting

$$\mu = \frac{1}{1+\alpha y} \tag{11}$$

into (6) and (7) and performing the integrations, the velocity profiles are

$$u\,(y) = \Lambda y \left(\alpha \frac{y^2}{3} + \frac{3-\alpha^2}{3(2+\alpha)} y - \frac{3+2\alpha}{3(2+\alpha)} \right). \tag{12}$$

The velocity profiles are shown for various α parameters in Fig. 2.

The velocity profile for $\alpha = 0$ corresponds to the constant viscosity case which is symmetric with respect to y. As α increases, the velocities are higher due to reduced viscosity and the profiles are not symmetric with the maximum velocities occurring for $y > 0.5$.

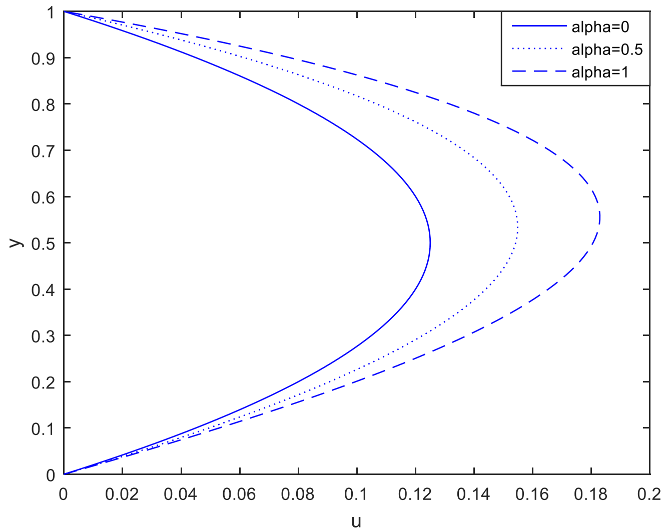

Fig. 2: Velocity profiles for $\mu = \dfrac{1}{1+\alpha y}$ $(\Lambda = -1)$.

The discharge is $Q = \int_0^1 u dy$, or

$$Q = -\frac{\Lambda}{36}\frac{\alpha^2 + 6\alpha + 6}{2+\alpha}\,, \tag{13}$$

which is a continuously increasing function in the domain of $\alpha \in (-2, \infty)$.

3.3 $\mu = 1 + \alpha y^2$ CASE

A similar functional relationship was taken in [13]. In this case, the viscosity of the fluid changes with a quadratic function of the spatial variable. Taking

$$\mu = 1 + \alpha y^2 \tag{14}$$

in the integrals (6) and (7), the velocity profiles turn out to be

$$u\,(y) = \frac{\Lambda}{2\alpha}\left(\ln\left(1+\alpha y^2\right) - \frac{\ln\left(1+\alpha\right)}{\arctan\sqrt{\alpha}}\arctan\sqrt{\alpha}y\right). \tag{15}$$

The corresponding velocity profiles are depicted in Fig. 3.

The velocity profiles are not symmetric. As α increases, the viscosity increases, thereby resulting in lower velocity profiles with the maximum velocities occurring for $y < 0.5$.

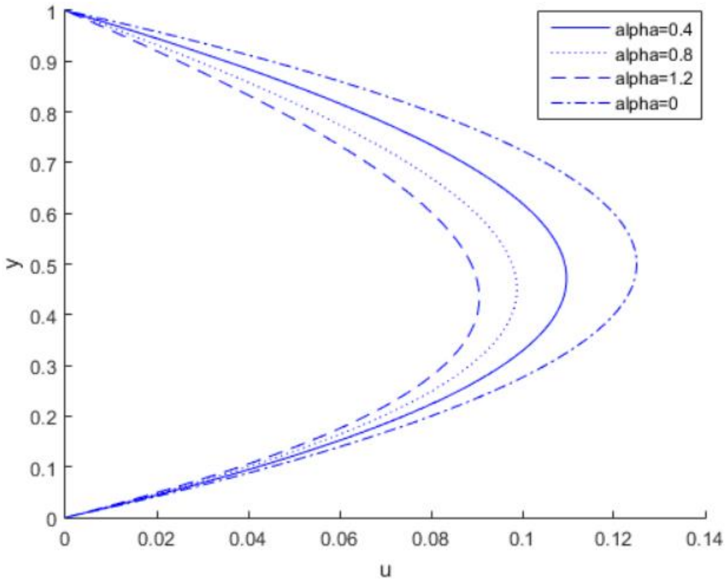

Fig. 3: Velocity profiles for $\mu = 1 + \alpha y^2$ $(\Lambda = -1)$.

The discharge is

$$Q = \frac{\Lambda}{4\alpha\sqrt{\alpha}} \frac{4\text{arctan}^2\sqrt{\alpha} + \ln^2(1+\alpha) - 4\sqrt{\alpha}\text{arctan}\sqrt{\alpha}}{\text{arctan}\sqrt{\alpha}}, \tag{16}$$

which is a decreasing function for positive values of α.

3.4 $\mu = e^{\alpha y}$ CASE

Power law variations, hyperbolic sine variations which can be approximated by exponential variations were already used in the literature [11, 12, 14]. Direct exponential type variations were also considered [16]. In this case, the viscosity of the fluid changes with the functional form

$$\mu = e^{\alpha y}. \tag{17}$$

The velocity profiles are

$$u(y) = \frac{\Lambda}{\alpha}\left(\frac{e^{-\alpha}}{e^{-\alpha}-1}\left(e^{-\alpha y}-1\right) - ye^{-\alpha y}\right) \tag{18}$$

also in exponential form. The corresponding velocity profiles are drawn in Fig. 4.

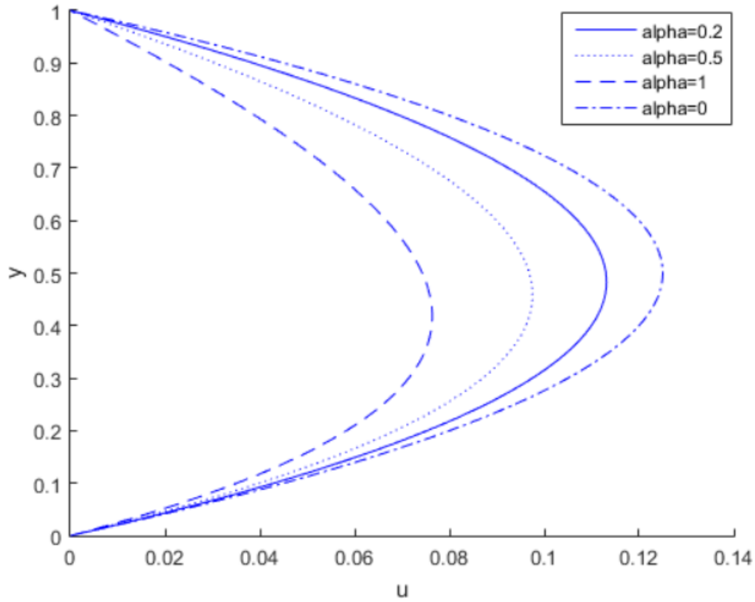

Fig. 4: Velocity profiles for $\mu = e^{\alpha y}$ $(\Lambda = -1)$.

The maximum velocities decrease when α increases. They occur at lower height values as α increases.

The discharge for this case is

$$Q = -\frac{\Lambda}{\alpha}\left(\frac{e^{-\alpha}}{e^{-\alpha}-1} - \frac{1}{\alpha^2}\left(e^{-\alpha}-1\right)\right), \tag{19}$$

which is again a decreasing function for positive values of α.

In conclusion, the velocity profiles, the maximum velocities and discharges alter much with the viscosity variation parameter in each model.

4 Discontinuous Viscosity Variations

In this section, two problems will be considered. In the first problem, parallel plate flow of two immiscible fluids with different viscosities will be considered. In the second problem, the high viscosity intermediate fluid is lubricated by a low viscosity fluid of thin film at the boundaries. To construct a uniform solution valid throughout the whole domain, the gamma interval function will be employed which is defined as

$$\gamma\,[a,\,b)\,(y) = \begin{cases} 1 & a \leq y < b \\ 0 & y < a,\; y \geq b \end{cases}, \tag{20}$$

which states that if the independent variable is within the limits of the range, it is 1 and 0 otherwise. This special function is new and used for the first time to construct a single analytical solution combining two different perturbation-iteration solutions of the Blasius equation [17]. The gamma interval function will enable us to write a compact solution for the whole range of domain.

4.1 Two different constant viscosity fluids

Assume that there are two different fluids laying on top of each other with the first fluid having dimensional viscosity μ_0 and corresponding dimensionless viscosity 1 and the other fluid having dimensional viscosity μ_1 and the corresponding dimensionless viscosity α $(\alpha = \mu_1/\mu_0)$. The dimensionless shear stress is

$$\tau = \frac{du}{dy}\gamma\left[0,\beta\right)(y) + \alpha\frac{du}{dy}\gamma[\beta,1]\left(y\right) . \tag{21}$$

In terms of the gamma interval functions, the expression clarifies that the first fluid with dimensionless viscosity 1 lays from $y = 0$ to $y = \beta$ and the next fluid with dimensionless viscosity α lays from $y = \beta$ to $y = 1$, where $0 < \beta < 1$.

To distinguish the solutions from each other, at this stage one may state the velocity profile of the first fluid by $u(y)$ and the second fluid by $v(y)$. Solving (4) for both fluids, the solutions are

$$u = \frac{\Lambda}{2}y^2 + c_1 y + c_2\,, \qquad v = \frac{\Lambda}{2\alpha}y^2 + c_3 y + c_4\,. \tag{22}$$

The boundary conditions require no slip condition at the boundaries and equivalency of velocity and shear stresses at the contact location

$$u(0) = 0\,, \qquad v(1) = 0\,, \qquad u\left(\beta\right) = v(\beta)\,, \qquad \frac{du}{dy}\left(\beta\right) = \alpha\frac{dv}{dy}(\beta)\,. \tag{23}$$

Applying the boundary conditions, the results for each fluid are

$$u(y) = \frac{\Lambda}{2}\left(y^2 + \frac{\beta^2\left(\alpha-1\right)+1}{\beta\left(1-\alpha\right)-1}y\right)\,, \tag{24}$$

$$v(y) = \frac{\Lambda}{2\alpha}\left(y^2 + \frac{\beta^2\left(\alpha-1\right)+1}{\beta\left(1-\alpha\right)-1}y - \frac{\beta(1-\alpha)(1-\beta)}{\beta\left(1-\alpha\right)-1}\right)\,. \tag{25}$$

One can unite the two solutions and express them as a single velocity function $U\left(y\right)$ in terms of the gamma interval functions

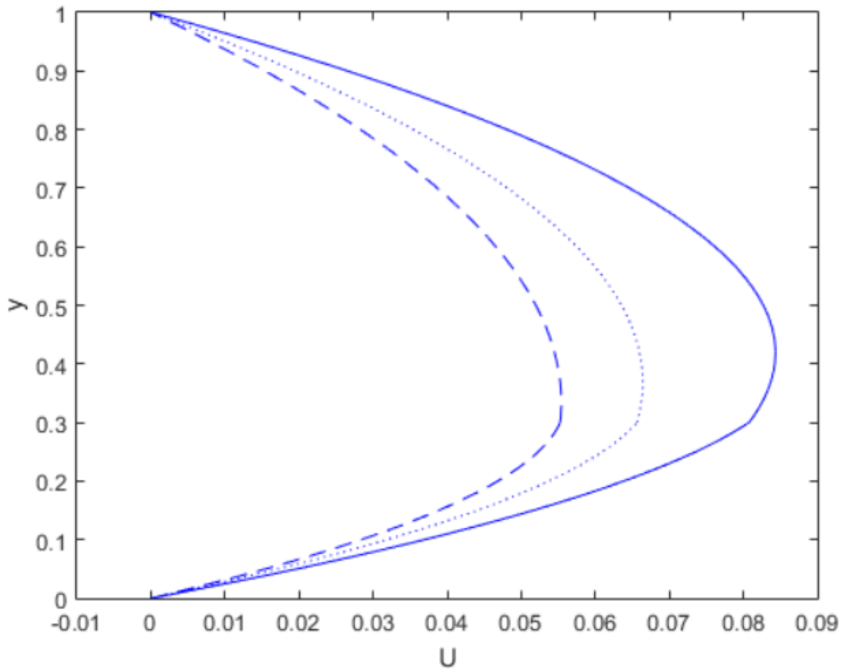

Fig. 5: Velocity profiles for various α values: $\alpha = 2$ (solid); $\alpha = 3$ (dotted); $\alpha = 4$ (dashed) ($\Lambda = -1$, $\beta = 0.3$).

$$U(y) = \frac{\Lambda}{2}\Big(y^2 + \frac{\beta^2(\alpha-1)+1}{\beta(1-\alpha)-1}y\Big)\gamma[0,\beta)(y) + \frac{\Lambda}{2\alpha}\Big(y^2 + \frac{\beta^2(\alpha-1)+1}{\beta(1-\alpha)-1}y - \frac{\beta(1-\alpha)(1-\beta)}{\beta(1-\alpha)-1}\Big)\gamma[\beta,1](y)\,. \tag{26}$$

The velocity profiles are given for various α values in Figure 5.

The two fluids meet at $\beta = 0.3$. As the relative viscosity of the upper fluid increases, the velocities decrease due to the combine effect of both fluids. The velocities and shear stresses are equal at the intermediate point but the slopes are different.

Figure 6 depicts the variations due to the change in the intermediate point.

Generally speaking, as β increases, the fluid with less viscosity dominates the region thereby increasing the velocities and discharges.

The discharge is

$$Q = \frac{\Lambda}{12\alpha}\frac{1+(1-\alpha)(-4\beta+6\beta^2-4\beta^3+\beta^4(1-\alpha))}{\beta(1-\alpha)-1}\,. \tag{27}$$

If $\alpha = 1$, then $Q = -\dfrac{\Lambda}{12}$, the homogenous discharge is retrieved.

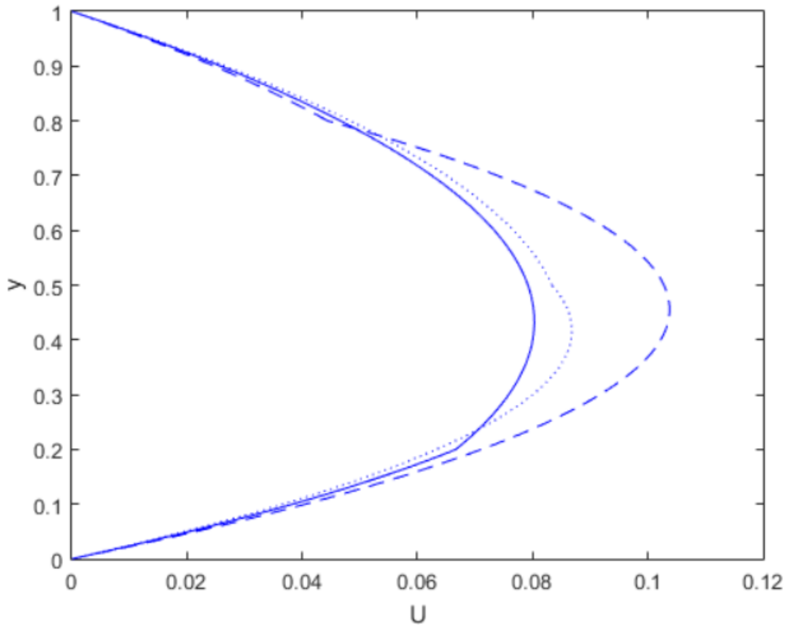

Fig. 6: Velocity profiles for various β values: $\beta = 0.2$ (solid); $\beta = 0.5$ (dotted); $\beta = 0.8$ (dashed) ($\Lambda = -1$, $\alpha = 2$).

4.2 AN INTERMEDIATE HIGH VISCOSITY FLUID LUBRICATED WITH A LOW VISCOSITY FLUID

To take the advantage of the symmetry, the dimensionless coordinates are chosen slightly different from the previous problems, that is $y = 0$ corresponds to the symmetric midline between the parallel plates and $y^* = \mp h$ or $y = \mp 1$ corresponds to the boundaries as shown in Fig. 7. There is a low viscosity flow adjacent to the boundaries and a high viscosity flow in the region $-\beta < y < \beta$. Note that $1-\beta \ll \beta$ for practical lubrication purposes.

It is sufficient to solve the positive part of the problem due to symmetry. The lower viscosity fluid u is adjacent to the boundaries with $\mu = 1$ and the higher viscosity fluid v is in the interior part with $\mu = \alpha$ (Fig. 7). The solutions are

$$u = \frac{\Lambda}{2} y^2 + c_1 y + c_2 \,, \qquad v = \frac{\Lambda}{2\alpha} y^2 + c_3 y + c_4 \,. \tag{28}$$

The boundary conditions require

$$\frac{dv}{dy}(0) = 0 \,, \qquad u(1) = 0 \,, \qquad u(\beta) = v(\beta) \,, \qquad \frac{du}{dy}(\beta) = \alpha \frac{dv}{dy}(\beta) \,, \tag{29}$$

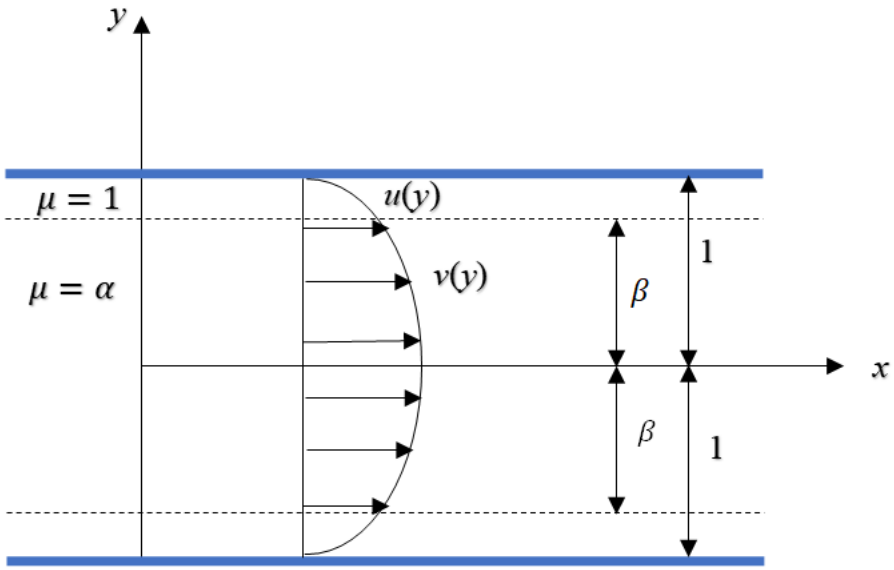

Fig. 7: Dimensionless symmetrical flow geometry of the problem.

where the first condition is the symmetry condition, the second is the no-slip condition, the third and fourth conditions are the equivalence of the velocities and shear stresses at the contact lines of the fluids. Applying the boundary conditions, the results for each fluid are

(30) $$u\left(y\right)=\frac{\Lambda}{2}\left(y^{2}-1\right),$$

(31) $$v\left(y\right)=\frac{\Lambda}{2\alpha}\left(y^{2}+\beta^{2}\left(\alpha-1\right)-\alpha\right)\ .$$

One can unite the two solutions and express them as a single velocity function $U\left(y\right)$ in terms of the gamma interval functions

(32) $$U\left(y\right)=\frac{\Lambda}{2\alpha}\left(y^{2}+\beta^{2}\left(\alpha-1\right)-\alpha\right)\gamma\left[0,\beta\right)\left(y\right)+\frac{\Lambda}{2}\left(y^{2}-1\right)\gamma[\beta,1]\left(y\right),$$

which are the velocity profiles for the positive y axis. The velocity profiles are given for various α values in Fig. 8.

As the relative viscosity of the intermediate fluid increases, the velocities profiles decrease. The velocities and shear stresses are equal at the contact points but the slopes are different.

The discharge is

(33) $$Q=-\frac{2\Lambda}{3\alpha}\left(\beta^{3}\left(1-\alpha\right)+\alpha\right)\ .$$

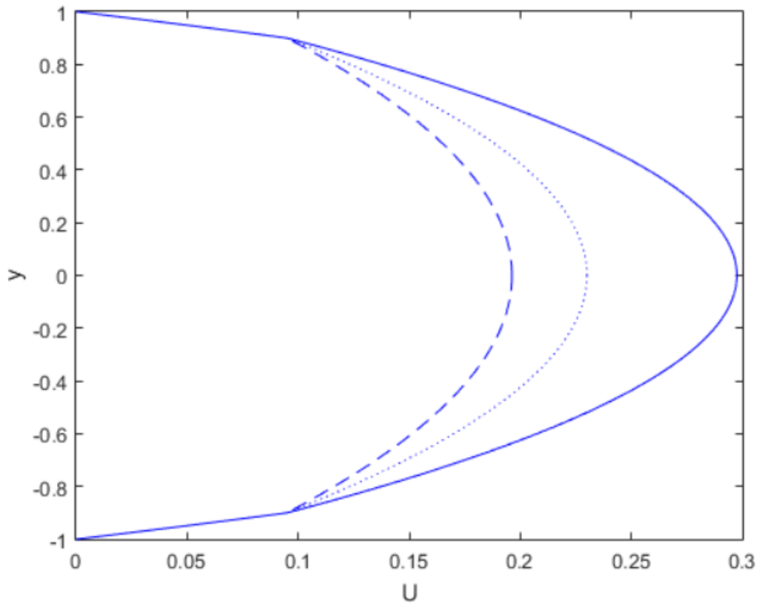

Fig. 8: Velocity profiles for various α values: $\alpha = 2$ (solid); $\alpha = 3$ (dotted); $\alpha = 4$ (dashed) ($\Lambda = -1$, $\beta = 0.9$).

If $\beta \to 1$, the space is filled with only the high viscosity fluid and the discharge would be $Q = -\dfrac{2\Lambda}{3\alpha}$ which is the Newtonian solution for a single fluid. Taking the ratio of the lubricated discharge to the unlubricated discharge containing only the high viscosity fluid, one has

$$R_Q = \alpha\left(1 - \beta^3\right) + \beta^3\,. \tag{34}$$

The intermediate fluid to be conveyed is highly viscous and hence $\alpha > 1$. Since $\beta < 1$, the ratio is always bigger than 1, i.e. $R_Q > 1$ and linearly increases with α for a given fixed β. This shows the importance of conveying high viscosity fluids with a lubrication low viscosity fluid at the boundaries. Petroleum is sent easier with such a lubrication with water. In calculating the ratio in (34), the discharge due to the low viscosity fluid is also taken into account. A more precise ratio would be to compare the discharges of high viscosity fluids only

$$\bar{R}_Q = \frac{3}{2}\beta\left(1 - \beta^2\right)\alpha + \beta^3\,. \tag{35}$$

For $\alpha > 1$, this ratio is also greater than 1, i.e. $\bar{R}_Q > 1$, increases linearly with α, justifying the need for lubrication of a high viscosity fluid to increase discharges.

5 Concluding Remarks

Parallel plate flow of Newtonian fluids with spatial dependent viscosities are considered. Exact analytical solutions are given for four different cases of continuous variation in the viscosity. The velocity profiles and discharges are given for each case considered. For the discontinuous case, two problems are considered. In the first problem, the immiscible two different fluid flow case laying on top of each other is considered. Then as a next problem, the lubrication of a highly viscous intermediate fluid with low viscosity fluid adjacent to the boundaries is studied. It is found that viscosity variations may have substantial effects on the velocities and discharges and cannot be neglected for precise solutions.

The analysis presented here may be extended to pipe flows with continuous and discrete viscous variation in the radial direction. For pipe flow of unsymmetrical velocity profiles with respect to the symmetry axis along the length, solutions might be much more complex requiring two variables namely the radial and angular coordinates within the cross- section to represent the motion. Another extension might be to use a Newtonian and a non-Newtonian fluid which are immiscible to calculate the velocity profiles and discharges.

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