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GENERAL MECHANICS

REGIONS OF EXISTENCE OF ANALYTICAL SOLUTIONS OF THE (2+1)-DIMENSIONAL SINE-GORDON EQUATION

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ABSTRACT: The sine-Gordon equation is a well-known equation, finding application in a multitude of physical phenomena. In this paper we consider the two-dimensional sine-Gordon equation and its Jacobi elliptic function solutions. We find and illustrate the regions of existence for the solutions, expanding on equivalences found between different solutions. Applying the regions of existence in the setting of band two-dimensional Josephson junctions, a different kind of relation appears between the amplitude of the wave and the width of the junction. Exploring this relation, we give a concrete example of the influence of the wave amplitude on the magnetic field in the junction.

KEY WORDS: analytical solution, high-amplitude waves, Jacobi elliptic functions, Josephson junction, magnetic field, regions of existence, sine-Gordon equation.

1 INTRODUCTION

The one dimensional sine-Gordon equation (SGE)

$$u_{xx} - u_{tt} = \sin(u)$$

is an important nonlinear partial differential equation (PDE). It first originated in differential geometry in 1862 [1] and was later rediscovered in 1939 in the study of crystal dislocations [2]. The surge of interest in this equation is due to it having special solutions, e.g. solitons and breathers [3, 4].

It has a wealth of applications – the description of self-induced transperancy [5], the dynamics in long Josephson junctions [6, 7], the description of magnetic domain wall dynamics [8], and others [4, 9, 10].

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This PDE also has plenty of generalisations – the sinh-Gordon equation [11, 12], the elliptic sine-Gordon equation [11], the quantum sine-Gordon model [13, 14], the stochascit or dynamical sine-Gordon equation [15], the supersymmetric sine-Gordon model [16, 17] being some of the most well-known.

In 1992 Martinov and Vitanov published an article [18], describing a procedure for solving the (2+1)-dimensional SGE. In a series of articles [19–21] they delve deeper into the results, clarifying and expanding on the original paper. While providing a plethora of new solutions, the relationship between them and the constraints on their parameters has only partially been investigated.

In this paper, we will use the technique, realised in [18], to elucidate this relationship and to group equivalent solutions, found through this method.

2 An Overview of the known Jacobi Elliptic Function Solutions

Let us first go over the procedure. To solve the two-dimensional SGE

$$u_{xx} + u_{yy} - u_{tt} = \sin[u(x, y, t)] \tag{1}$$

we use the ansatz

$$u(x, y, t) = 4 \arctan[Af(\alpha x; k_1)g(\beta y + \gamma t; k_2)], \tag{2}$$

where A, α, β and γ are real parameters, f and g are real Jacobi elliptic functions and k_1 and k_2 are their modules with the following constraints:

$$\begin{aligned} 0 \leq k_1 \leq 1 \\ 0 \leq k_2 \leq 1. \end{aligned} \tag{3}$$

For brevity, we introduce the notation

$$(f, g) = 4 \arctan(Af(\alpha x; k_1)g(\beta y + \gamma t; k_2)).$$

Using the general equations of the Jacobi elliptic functions:

$$\begin{aligned} (f')^2 &= a_1 f^4 + b_1 f^2 + c_1 \\ (g')^2 &= a_2 g^4 + b_2 g^2 + c_2, \end{aligned}$$

where $a_i, b_i, c_i, i = 1, 2$ are parameters, and substituting into (2), we get

$$\begin{aligned} 2f^3 g[\alpha^3 a_1 - (\beta^2 - \gamma^2)A^2 c_2] + fg[\alpha^2 b_1 + (\beta^2 - \gamma^2)b_2 - 1] \\ + 2fg^3[(\beta^2 - \gamma^2)a_2 - \alpha^2 A^2 c_1] \\ + f^3 g^3[A^2 - \alpha^2 A^2 b_1 - (\beta^2 - \gamma^2)A^2 b_2] = 0. \end{aligned}$$

By extracting the coefficients in front of the products of f and g and setting them to zero, we get the following system:

$$(4)\qquad \left| \begin{aligned} a_1 &= -\frac{A^2(\gamma^2-\beta^2)}{\alpha^2}c_2 \\ b_1 &= \frac{1+(\gamma^2-\beta^2)b_2}{\alpha^2} \\ c_1 &= -\frac{(\gamma^2-\beta^2)}{\alpha^2A^2}a_2. \end{aligned} \right.$$

Solving this system would mean solving the partial differential equation. In [19], three classes of solutions are discussed, which depend on the sign of $\gamma^2 - \beta^2$. We will restrict our analysis on the case $\gamma^2 - \beta^2 > 0$. For convenience, we will use $n^2 = \gamma^2 - \beta^2$.

Seeing as we have 12 classic Jacobi functions, we could potentially have 144 different solutions. However, due to the well known form of a_i, b_i and c_i [22] for the Jacobi elliptic functions, we can narrow down out search. For example, $(\mathrm{cn}, \mathrm{sn})$ has $c_1 = 1-k_1^2 > 0$ and $a_2 = k_2^2 > 0$ and, comparing signs in (4), we get a contradiction. However, for $(\mathrm{cn}, \mathrm{cn})$ we get no such incompatibilities.

Another way to reduce the number of potential solutions is to notice that if $b_2 > 0$, then necessarily $b_1 > 0$. For instance, looking at the pair $(\mathrm{sn}, \mathrm{dn})$ that passes the previous check, we see that $b_1 = -(1 + k_1^2) < 0$ and $b_2 = (2 - k_2^2) > 0$. Therefore, $(\mathrm{sn}, \mathrm{dn})$ is not a valid possibility.

Exploiting these relations, we can narrow the potential solutions to 24 pairs. However, many of those pairs are equivalent. In [18] there are 7 unique solutions found.

3 Regions of Existence

We will present the regions of existence of the solutions in the order in which they are presented in [18]. The modules will differ from the original paper, but only in form for easier analysis of the solutions. In addition, not all region plots will be with the same scale, due to some regions of existence being more restrictive. Due to A always appearing squared in the regions, we show plots only for $A > 0$, the other case yeilding symmetric results.

1. $(\mathrm{cn}, \mathrm{cn})$ – here we have

$$\begin{aligned} k_1^2 &= \frac{A^4 + A^2(1+A^2)n^2}{A^4 - 1 + (1+A^2)^2n^2}, \\ k_2^2 &= \frac{A^2((1+A^2)n^2 - 1)}{(1+A^2)^2n^2}, \\ \alpha^2 &= 1 - \frac{2}{1+A^2} + n^2. \end{aligned}$$

In order for the solution to be valid, we have to take into account (3) and we add the constraint that $\alpha^2 \geq 0$. Solving the system

$$\left| \begin{array}{l} 0 \leq k_1^2 \leq 1 \\ 0 \leq k_2^2 \leq 1 \\ \alpha^2 \geq 0, \end{array} \right.$$

we find the region of existence (Fig. 1).

$$\mathcal{D}_1 = \left\{ (n^2, A) \in \mathbb{R}^2 : A \in \mathbb{R} \text{ and } n^2 \geq \frac{1}{1+A^2} \right\}.$$

Let us note that although $A = 0$ is allowed, it leads to the trivial solution $u(x, y, t) = 0$.

2. $(\mathrm{dn}, \mathrm{sn})$ – in this case we get

$$k_1^2 = \frac{(1+A^2)^2 n^2 - 1}{A^2(1+A^2)n^2},$$
$$k_2^2 = \frac{A^2 - A^2(1+A^2)n^2}{(1+A^2)n^2},$$
$$\alpha^2 = A^2 n^2.$$

The region of existence is (Fig. 2)

$$\mathcal{D}_2 = \left\{ (n^2, A) \in \mathbb{R}^2 : \begin{array}{ll} \frac{A^2}{(1+A^2)^2} \leq n^2 \leq \frac{1}{1+A^2}, & |A| \geq 1 \\ \frac{1}{(1+A^2)^2} \leq n^2 \leq \frac{1}{1+A^2}, & |A| < 1 \end{array} \right\}.$$

In the search for solutions, one can also find a solutions of the form $(\mathrm{dn}, \mathrm{cd})$, $(\mathrm{nd}, \mathrm{sn})$ and $(\mathrm{nc}, \mathrm{cn})$. These solutions are equivalent to $(\mathrm{dn}, \mathrm{sn})$ in the following sense – under the transformations [22]:

$$\mathrm{cd}(u; k) = \mathrm{sn}(u + K(k); k), \tag{5}$$

where $K(k)$ is the full elliptic integral of the first kind, and

$$\mathrm{nd}(u; k) = \mathrm{dn}\left(k'u; i\frac{k}{k'}\right), \tag{6}$$

where, $k' = \sqrt{1-k^2}$, the parameters k_1, k_2 and α for $(\mathrm{dn}, \mathrm{sn})$ and the other functions are equal. We will denote this equivalence by $(\mathrm{dn}, \mathrm{sn}) \sim (\mathrm{dn}, \mathrm{cd})$. Therefore, we have $(\mathrm{dn}, \mathrm{sn}) \sim (\mathrm{dn}, \mathrm{cd}) \sim (\mathrm{nd}, \mathrm{sn}) \sim (\mathrm{nd}, \mathrm{cd})$.

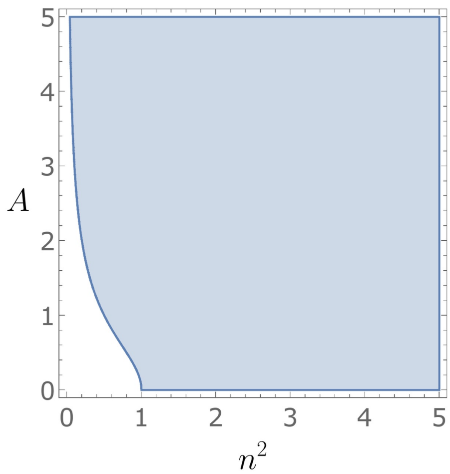

Fig. 1. Region of existence for $(\mathrm{cn}, \mathrm{cn})$.

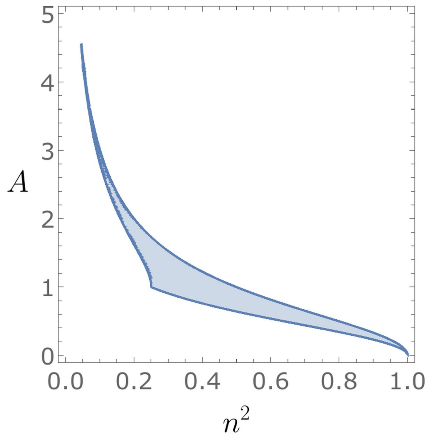

Fig. 2. Region of existence for $(\mathrm{dn}, \mathrm{sn})$.

3. $(\mathrm{dn}, \mathrm{sc})$ – for this solution we have

$$\begin{aligned} k_1^2 &= \frac{1+(A^2-1)^2 n^2}{A^2(A^2-1)n^2}, \\ k_2^2 &= \frac{A^2-(A^2-1)^2 n^2}{(A^2-1)n^2}, \\ \alpha^2 &= A^2 n^2. \end{aligned}$$

The region of existence is (Fig. 3)

$$\mathcal{D}_3 = \left\{ (n^2, A) \in \mathbb{R}^2 : |A| > 1 \text{ and } \frac{1}{A^2-1} \le n^2 \le \frac{A^2}{(A^2-1)^2} \right\}.$$

By combining the transformations [22]

$$\begin{aligned} \mathrm{sn}(u+iK(k);k) &= \frac{1}{k}\mathrm{sn}(u;k) \\ \mathrm{cn}(u+iK(k);k) &= -\frac{i}{k}\mathrm{ds}(u;k) \end{aligned}$$

we get

$$\mathrm{sc}(u+iK(k);k) = i\mathrm{nd}(u;k). \tag{7}$$

Using this transformation twice results in $(\mathrm{dn}, \mathrm{sc}) \sim (\mathrm{cs}, \mathrm{nd})$.

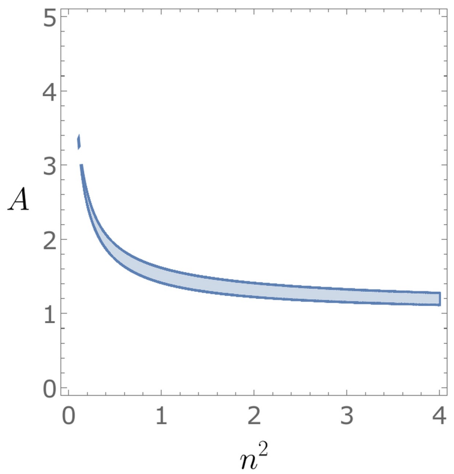

Fig. 3. Region of existence for $(\mathrm{dn}, \mathrm{sc})$.

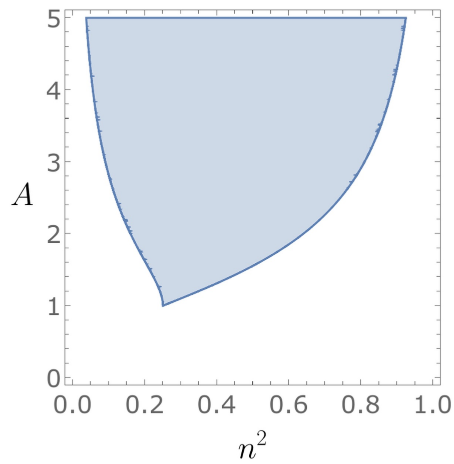

Fig. 4. Region of existence for $(\mathrm{dn}, \mathrm{ns})$.

4. $(\mathrm{dn}, \mathrm{ns})$ – here we find

$$k_1^2 = \frac{(A^2+1)^2 n^2 - A^4}{A^2(A^2+1)n^2 - A^4},$$
$$k_2^2 = \frac{A^2 - (A^2+1)n^2}{A^2(A^2+1)n^2},$$
$$\alpha^2 = \frac{A^2}{1+A^2} - n^2.$$

The region of existence is (Fig. 4)

$$\mathcal{D}_4 = \left\{ (n^2, A) \in \mathbb{R}^2 : |A| \geq 1 \text{ and } \frac{A^2}{(A^2+1)^2} \leq n^2 \leq \frac{A^4}{(A^2+1)^2} \right\}.$$

Using transformations (5) and (6) results in $(\mathrm{dn}, \mathrm{ns}) \sim (\mathrm{dn}, \mathrm{dc}) \sim (\mathrm{nd}, \mathrm{ns}) \sim (\mathrm{nd}, \mathrm{dc})$.

5. $(\mathrm{dn}, \mathrm{cs})$ – for this solution

$$k_1^2 = \frac{A^4 + (A^2-1)^2 n^2}{A^4 + A^2(A^2-1)n^2},$$
$$k_2^2 = \frac{(A^2-1)^2 n^2 - A^2}{A^2(A^2-1)n^2},$$
$$\alpha^2 = \frac{A^2}{A^2-1} + n^2.$$

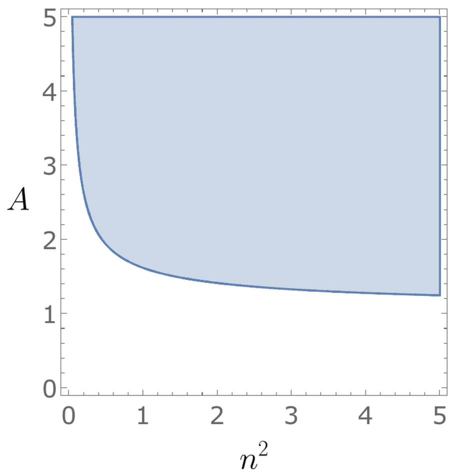

Fig. 5. Region of existence for $(\mathrm{dn}, \mathrm{cs})$.

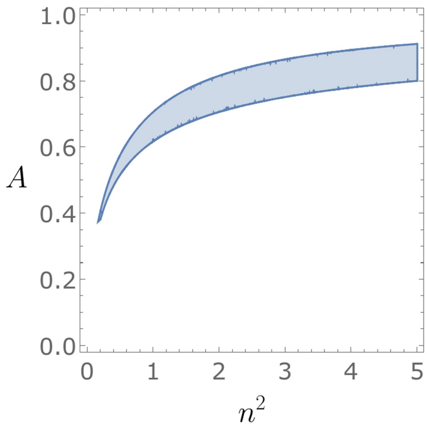

Fig. 6. Region of existence for $(\mathrm{sc}, \mathrm{dn})$.

The region of existence is (Fig. 5)

$$\mathcal{D}_5 = \left\{ (n^2, A) \in \mathbb{R}^2 : |A| > 1 \text{ and } n^2 \geq \frac{A^2}{(A^2-1)^2} \right\}.$$

Using the transformation (7) results in $(\mathrm{dn}, \mathrm{cs}) \sim (\mathrm{cs}, \mathrm{dn})$.

6. $(\mathrm{sc}, \mathrm{dn})$ – here we get

$$k_1^2 = \frac{A^4 + (1-A^2)^2 n^2}{(1-A^2)n^2},$$
$$k_2^2 = \frac{A^2 - (1-A^2)^2 n^2}{A^2(1-A^2)n^2},$$
$$\alpha^2 = \frac{n^2}{A^2}.$$

The region of existence is (Fig. 6)

$$\mathcal{D}_6 = \left\{ (n^2, A) \in \mathbb{R}^2 : |A| < 1 \text{ and } \frac{A^2}{1-A^2} \leq n^2 \leq \frac{A^2}{(1-A^2)^2} \right\} \setminus \{(0,0)\}.$$

Using the transformation (7) results in $(\mathrm{sc}, \mathrm{dn}) \sim (\mathrm{nd}, \mathrm{cs})$.

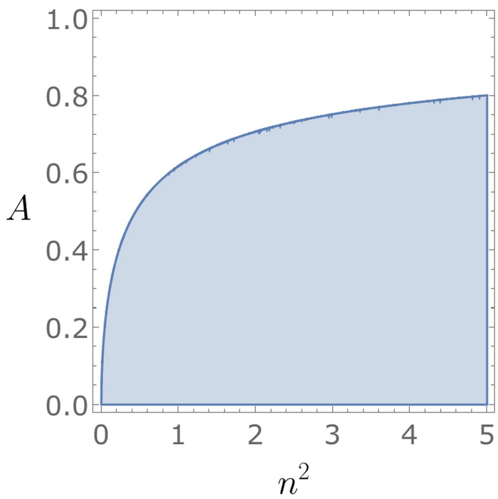

Fig. 7. Region of existence for $(\mathrm{sc}, \mathrm{nd})$.

7. $(\mathrm{sc}, \mathrm{nd})$ – in this final solution we get

$$k_1^2 = \frac{1+(1-A^2)^2 n^2}{1+(1-A^2)n^2},$$
$$k_2^2 = \frac{(1-A^2)^2 n^2 - A^2}{(1-A^2)n^2},$$
$$\alpha^2 = \frac{1}{1-A^2} + n^2.$$

The region of existence is (Fig. 7)

$$\mathcal{D}_7 = \left\{ (n^2, A) \in \mathbb{R}^2 : |A| < 1 \text{ and } n^2 \geq \frac{A^2}{(1-A^2)^2} \right\}.$$

Again, $A = 0$ is allowed, but leads to the trivial solution $u(x, y, t) = 0$.

Using the transformation (7) results in $(\mathrm{sc}, \mathrm{nd}) \sim (\mathrm{nd}, \mathrm{sc})$.

4 The Two-Dimensional Josephson Junction

A Josephson junction is a system of weakly linked superconductors. The weak link could be a thin insulating barrier (superconductor-insulator-superconductor, or SIS, non-superconducting metal (SNS) or a constriction that weakens the superconductivity at the contact point or a weaker superconductor (SS'S) [23]. We will examine the SIS Josephson junction when the dissipative effects are negligible.

Following the example in [7], let the junction be made of the same superconducting plates, the thin dielectric layer be parallel to the O_{xy} plane with size l and dielectric constant ε. Let us denote $d = l + 2\lambda$, where λ is the London penetration depth. Furthermore, let $\mathbf{n} = (0, 0, 1)$, $\mathbf{I}_m$ be the Josephson current amplitude, u be the phase difference between the wave functions of the electrons in the superconductors and e, c, V be the electron charge, the speed of light and the applied voltage. Using Josephson's [24, 25] and Maxwell's equations [7]:

$$\begin{aligned}
&\mathbf{I} = \mathbf{I}_m \sin(u) \\
&u_{\hat{t}} = \frac{2eV}{\hbar} \\
&\nabla u = \frac{2ed}{\hbar c}\mathbf{H} \times \mathbf{n} \\
&\nabla \times \mathbf{E} = -\frac{1}{c}\mathbf{H}_{\hat{t}} \\
&\nabla \times \mathbf{H} = \frac{4\pi}{c}\mathbf{I} + \frac{\varepsilon}{c}\mathbf{E}_{\hat{t}}
\end{aligned}$$

Supposing that the Josephson current amplitude $\mathbf{I}_m$ and the electric field $\mathbf{E}$ are directed along the axis of the normal vector $\mathbf{n}$, while the magnetic field $\mathbf{H}$ propagates in the O_{xy} plane, that is $\mathbf{I}_m = (0, 0, i_m), \mathbf{H} = (H_1, H_2, 0)$ and $\mathbf{E} = (0, 0, E_3)$, we can reduce the equations to

$$u_{\hat{x}\hat{x}} + u_{\hat{y}\hat{y}} - \frac{1}{v_s^2}u_{\hat{t}\hat{t}} = \frac{1}{\lambda_J^2}\sin(u),$$

where $v_s = \lambda_J\omega_J$ is the electromagnetic wave velocity in the dielectric without Josephson current, $\lambda_J = \sqrt{\dfrac{\hbar c^2}{8\pi e d i_m}}$ is the Josephson length and $\omega_J = \sqrt{\dfrac{8\pi e d i_m}{\hbar\varepsilon}}$ is the Josephson plasma frequency.

After scaling the variables $x = \dfrac{\hat{x}}{\lambda_J}$, $y = \dfrac{\hat{y}}{\lambda_J}$ and $t = \hat{t}\omega_J$, we are left with (1). For the electric and magnetic field we find

$$\begin{aligned}
&\mathbf{E} = (0, 0, \frac{\hbar\omega_J}{2ed}u_t) \\
&\mathbf{H} = (-\frac{\hbar c}{2ed\lambda_J}u_y, \frac{\hbar c}{2ed\lambda_J}u_x, 0).
\end{aligned}$$

We will consider the band two-dimensional Josephson junction. This type is characterized by having its size in the O_x direction r much smaller that in the O_y direction, yet it cannot be neglected.

We will use the boundary conditions [7, 26]

$$u_x|_{x=0,r} = 0 \iff H_2|_{x=0,r} = 0\,,$$

which fix the external magnetic field in the O_x direction. Furthermore, while the solution can be shifted by expressing any of the above solutions as

$$u(x, y, t) = 4\arctan[Af(\alpha(x - x_0); k_1)g(\beta(y - y_0) + \gamma(t - t_0); k_2)],$$

we will assume that $x_0 = y_0 = t_0 = 0$. This will make the calculations simpler and will not affect the studied results.

Even though this is the simplest case, there are certain solutions which cannot fulfill these conditions. Indeed, solutions 6 and 7 have sc as the function dependant on x and, upon differentiating, we are left with

$$u_x(x, y, t) = \frac{4A\alpha\mathrm{dc}(\alpha x; k_1)\mathrm{nc}(\alpha x; k_1)g(\beta y + \gamma t; k_2)}{1 + (A\mathrm{sc}(\alpha x; k_1)g(\beta y + \gamma t; k_2))^2},$$

where g is either dn or nd. Seeing as how dc and nc can never be zero when $x \in \mathbb{R}$, we conclude that neither boundary condition can be fulfilled.

Concerning the other five solutions, we get sn and dn in the numerator of the derivative. At $x = 0$ the condition is automatically satisfied due to $\mathrm{sn}(0; k) = 0$ for any $k \in [0, 1]$. At the other end of the junction, we must have $\mathrm{sn}(\alpha r; k_1) = 0$, due to dn being strictly positive with real arguments. That leads to the nonlinear equation $r\alpha = 2mK(k_1)$, where $K(k_1)$ is the full elliptic integral of the first kind and $m \in \mathbb{Z}$. We have continued the investigation with $m = 1$.

Due to the strong non-linearity of the equation, we solve it numerically with the following intention – we fix r and A and look for n that satisfies the equation using

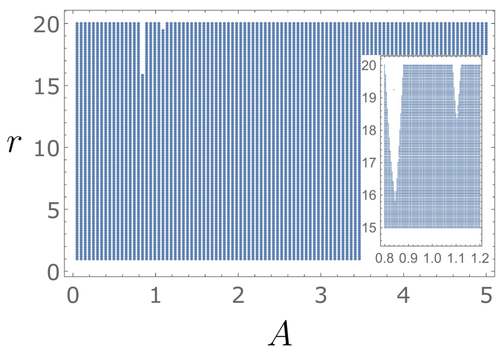

Fig. 8. AR plot for $(\mathrm{cn}, \mathrm{cn})$.

Newton's method. However, the method does not always converge on a real root. Even when we do find a real root, that solution may not be in the regions of existence of $u(x, y, t)$. Taking all of this into considerations, we create a different kind of region of existence – an AR plot that shows for which combinations of A and r we can find n such that $u(x, y, t)$ exists (Figs. 8-12).

These plots are made with $A \in [0.05, 5]$ or $A \in [1.05, 5]$, depending on the solution, $r \in [1, 20]$ and picking an initial condition $n_0 \in [0, 5]$ using evenly spaced points $N_A = 100, N_r = 100, N_{n_0} = 50$.

As we can see, for the solution with the least restricting region of existence $(\mathrm{cn}, \mathrm{cn})$ the AR plot is fittingly non-restrictive. For the other solutions with stricter conditions on the regions of existence, their respective AR plots are also stricter. On Figs. 9 and 10 we can see patterns of missing pairs emerging, taking approximately the same shape and enlarging with the amplitude of the wave A. As for the rest of the figures, a single shape is preserved, but stray lines of either missing (Fig. 11) or present pairs (Fig. 12) can be observed.

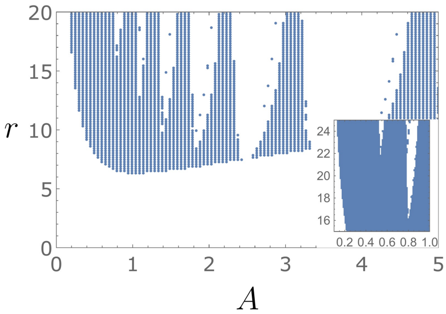

Fig. 9. AR plot for $(\mathrm{dn}, \mathrm{sn})$.

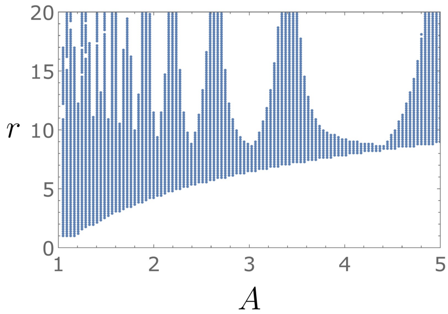

Fig. 10. AR plot for $(\mathrm{dn}, \mathrm{sc})$.

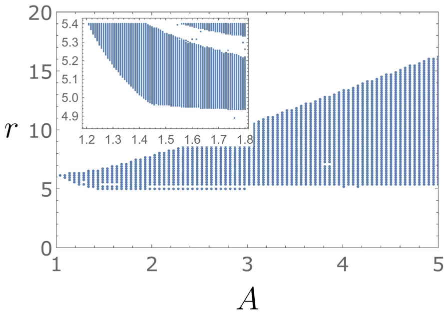

Fig. 11. AR plot for $(\mathrm{dn}, \mathrm{ns})$.

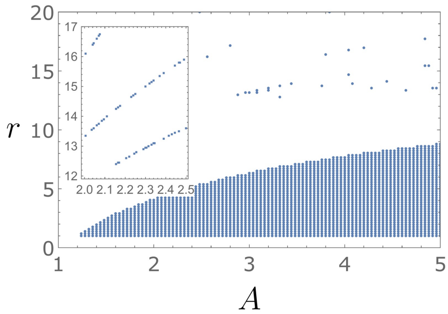

Fig. 12. AR plot for $(\mathrm{dn}, \mathrm{cs})$.

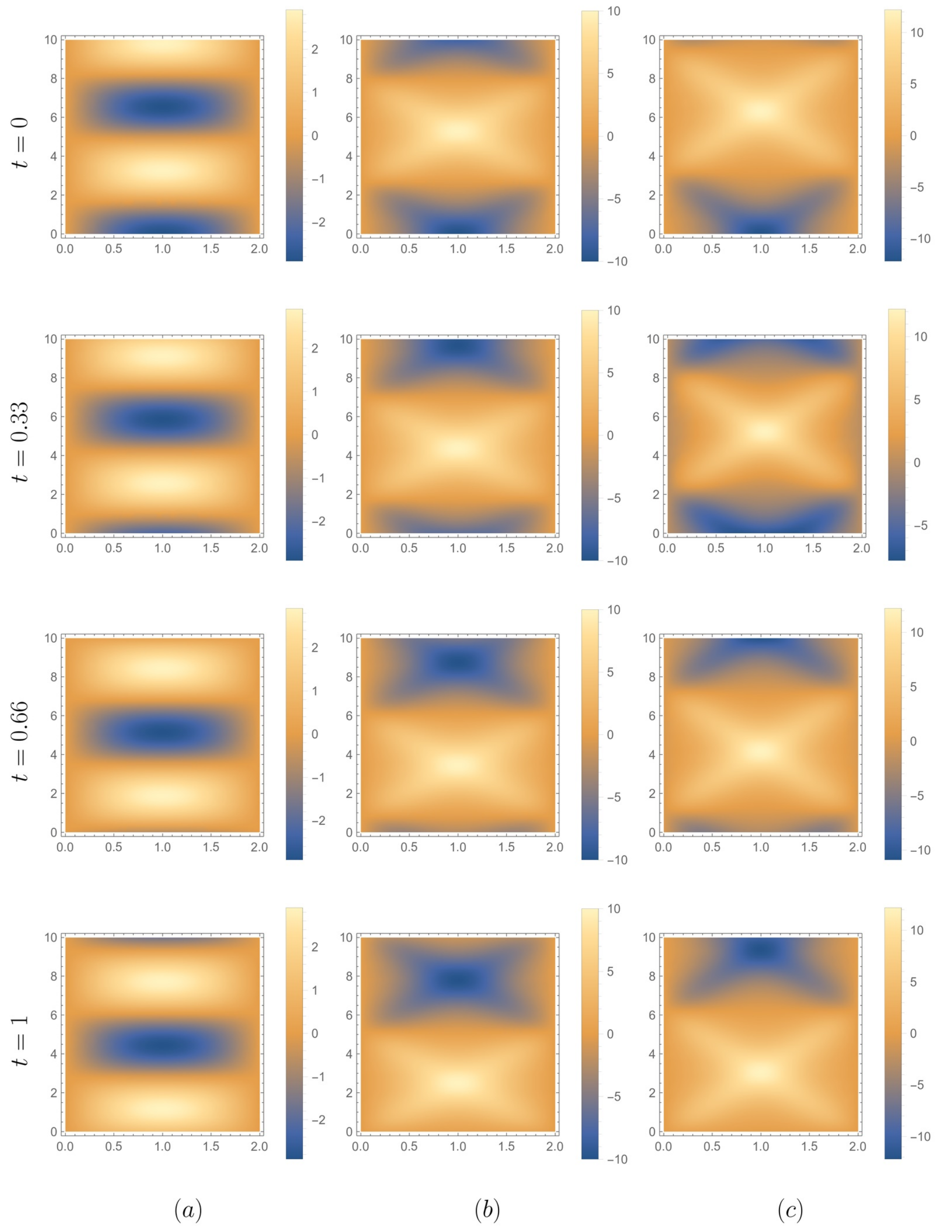

Fig. 13. u_x of $(\mathrm{cn}, \mathrm{cn})$ with $r = 2$, $A = 0.5, 3.5, 6$ and $n \approx 1.8582, 2.61471, 3.10512$ for (a), (b) and (c) respectively.

Based on these findings, one can investigate the effect of the width of the Josephson junction and the amplitude of the wave on the magnetic and electric fields only for pairs in the AR plots. As a demonstration, we consider u_x (which gives us information about the behavior of the y coordinate of the magnetic field up to a multiplication with a constant). We look into the solution $(\mathrm{cn}, \mathrm{cn})$, due to it allowing the most flexibility in terms of A and r, for $r = 2$. In this case a curious pattern emerges. The low-amplitude wave has its whole front rise and fall when moving in the y direction. As the amplitude increases, the front loses its unity. It separates into two peaks that pass trough each other while rising and falling in a soliton-like manner (Fig. 13).

5 Concluding remarks

In this paper, we have explored the regions of existence for the Jacobi elliptic function solutions, expanding on [18–21]. Using the proposed ansatz and sieving through the possible pairs of elliptic functions, we settle on 24 solutions, 7 of which have been discovered in [18] and others end up being equivalent to them.

Considering the two-dimensional Josephson junction, we look into the dependence on the width of the junction and the amplitude of the wave, providing AR plots for every solution that can fulfil the zero boundary conditions. We finish by giving an example of the effect of the wave amplitude on the magnetic field of the junction.

Needless to say, this paper only scratches the surface of the AR plots. Further numerical studies are required in order to fully grasp the relation between A and r. Once this has been done, a more informed and meaningful analysis of the magnetic and electric fields can be carried out.

More generally, a similar analysis could be conducted when the magnetic field is fixed in the O_y direction, instead of in the O_x direction, giving rise to different findings.

There are open questions with the analytical solutions as well. Of the 24 solutions, only 17 of them have been categorized. Currently, equivalences have not been found between the other 7 solutions and the known ones. Some of the regions of existence for these 7 solutions match the ones presented in this paper, while others have entirely different shapes. Even more curiously, the solution $(\mathrm{nc}, \mathrm{nc})$ is equivalent to $(\mathrm{cn}, \mathrm{nc})$ – a solution from the class $\gamma^2 - \beta^2 < 0$. For further investigations of the analytical solutions, there are promising leads and interesting results could be obtained.

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