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SPECIAL ISSUE

MODELING MUSICAL SYSTEMS USING MATLAB

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ABSTRACT: Musical systems for organizing tonal space are the foundation of music. Since Pythagoras, the topic has been a fruitful field for experiments for composers and music theorists. This is possible due to the correlation between human auditory perception specifics and how two or more frequencies relate to one another. The current approach towards modeling musical systems is focused on digitalizing the process for establishing the notes' frequencies and applying additive synthesis for designing timbre for the tone sets. Moreover, a possible application of the digital modeling process of the timbre for devising musical systems with a higher degree of intrinsic consonance in relation to the tonal system's specificities.

KEY WORDS: Musical systems, Tonal systems, Intrinsic dissonance, Musical modeling.

1 INTRODUCTION

Musical systems can be found in many aspects of music-making. Here the term will refer to and will be interchangeably used with the term tonal systems. Tonal systems have been accompanying music making since the dawn of music – they were the answer to the very important question – how to choose the proper fundamental frequency so that a musical scale can be built with them. Modeling tonal systems in MATLAB gives us flexible options to compare them theoretically and practically. This allows us to explore the authentic sound of different musical epochs and is an open field for creativity for designing the most suited systems for artistic expression. In the following sections, multiple MATLAB models of tonal systems will be reviewed and compared.

2 TONAL SYSTEMS

A tonal musical system describes the rules according to which frequencies are chosen so that a discrete basic set T with cardinality n (most commonly $n \in \{5, 7, 12, 19, \dots, n_{k-1} + n_k\}$ [1,2]) can be formed and then further augmented so that it can cover the

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practical tonal range – the keys of the piano (augmentation is done by multiplying and dividing the frequencies of the base set with powers of 2):

$$2^p, \quad p \in [-4; 4], \quad p \in Z.$$

The systems are divided into two general directions – intonations and temperaments.

2.1 INTONATIONS

The intonation systems use the relations between the row of the natural numbers (which coincides with the overtone series ratios) to form the building blocks of the octave. These systems are irregular and non-circulatory because they never temper any commas. They stay open and spiral upwards and downwards, because of this instead of having octave correspondence the scale has plentiful microchromatic versions of its degrees as long as the spiral extends upwards or downwards.

Also these systems' dimensionality depends on the number of generators and the prime overtones that are capable of representing (a.k.a. prime factor of the system):

- system with one generator, the perfect fifths (3/2), is linear and has a prime factor of 3 (Pythagorean systems) and has only one variant for the imperfect consonances;
- systems with two generators are 2-dimensional structures. They have generators just major thirds (5/4) and perfect fifths (3/2), have a prime limit of 5 (Syntonic systems), are 3-dimensional and have 2 microchromatic variations of the imperfect consonances;
- systems with 3 generators – fifths, thirds, and natural sevenths (7/4) are 3-dimensional, have a prime factor of 7 and 11 have three versions of the imperfect consonances.

The dimensionality grows as the value of the prime limit rises – with this also grows the number of microchromatic versions of the imperfect consonances that the system has.

2.2 TEMPERAMENTS

Temperaments are the other types of tonal systems that are in use. They are irrational in the way they generate the scale. In the general case an interval, most commonly the perfect octave, is divided into equal parts by using the irrational function of the n-th root, and from this discrete division of the octave a mapping to the 7 or 12 degrees of the scale is made such as to each interval between the scale degrees a discrete number of generators is designated. For example:

- in the 12 Equal divisions of the octave (EDO) system for each of the twelve steps of the octave one single generator is assigned;
- in the 72 EDO one step of the scale is mapped to six widths of the generator;
- in the 17 EDO one can choose a mapping that requires irregularity of the temperament – some steps will be mapped to one generator others to two or three – according to some aesthetical criteria.

2.3 12 EDO

The most common tonal system that is used in contemporary musical practice is the equal-tempered scale. It is a regular circulatory tonal system which frequencies can be derived by using the following formula:

$$P_n = P_a \, 2^{\frac{n-a}{12}} .$$

The indices refer to the desired pitch (n) and reference pitch (a) and the $2^{1/12}$ is the generator interval of the system (in this case it coincides with the width of the smallest step of the system's mapping to the 12 tone octave cardinality).

This system is very useful to model in MATLAB because it is a reference system for all other ones.

3 Modelling Historical Tonal Systems

Historica1tuning systems are systems that were once in practice. During more than two thousand years of music history in Western Europe there have been a myriad of different ways a keyboard is tuned – some had more impact on music making than others, others were local phenomena. Most of them were at one time an attempt to give an answer to the 4000-year-old question: how to stack 12 perfect fifths and get 7 perfect octaves? In reality, this is impossible because of the inequality

$$\left(\frac{3}{2}\right)^{12} > \left(\frac{2}{1}\right)^{7} .$$

The difference is well-known as the Pythagorean comma with ratio of 1.0136 or a width of 23.46 ¢ (one cent is 1200-th part of the width of a full octave 2/1).

Another comma that was historically important is the Syntonic comma – the comma of the 2-dimensional systems – the difference between the Pythagorean major third with a ratio of 81/64 and the just major third with a ratio of 5/4:

$$\left.\frac{81}{64}\middle/\frac{5}{4}\right. = \frac{81}{80}.$$

The interval ratio is 1.0125 or 21.5063 ¢.

Both commas are distributed between the circle of the fifths with some temperaments featuring distribution of the Pythagorean comma, other the Syntonic comma, like the following examples.

3.1 General temepraments for tuning a Hrpsichord

MATLAB was successfully used to model the following temperaments: Equal temperament, Open fifths temperament [3], Valotti, meantone for sharps, meantone for flats, Pythagorean, Grammateus, Kimberger II, Kirnberger III, Werkmeister, Corrette, Rameau and another version of the Grammateus called Twelve True Fifths.

Most of the modeling was done following charts provided by Carey Beebe in his web resources [4]. There he gave the diagram showing how the comma would be splitted and distributed (see Fig. 1).

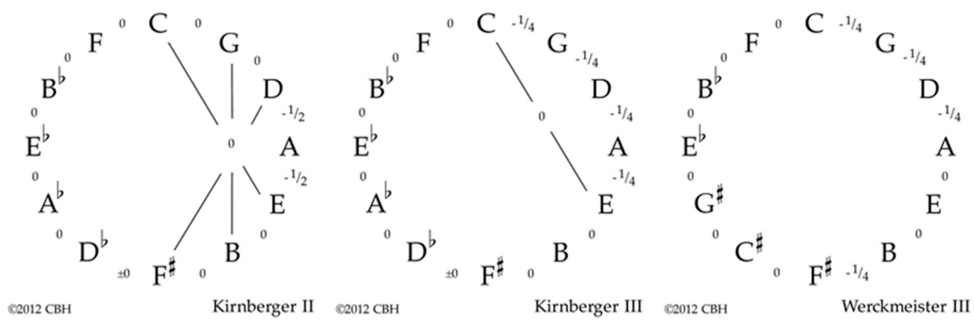


Fig. 1. Schemes for tuning historical temperaments [5–7].

On the scheme, one can clearly see the circle of fifths and the distribution of the commas between them. The schemes which have lines between thirds are required to use Syntonic comma as the other ones, with only fractions and zeroes between the fifths use the Pythagorean comma.

The resultant model is a table with the frequency for all the tones of the piano keyboard and is useful to show it as a surface plot (Figs. 2, 3, and 4).

```
>> vallotti(440,'a')

ans =

     27.50      29.23      30.80      32.81      34.65      36.75      38.98      41.16      43.85      46.20      49.11      51.97
     55.00      58.47      61.60      65.63      69.30      73.50      77.96      82.31      87.70      92.39      98.22     103.94
    110.00     116.94     123.19     131.26     138.59     147.00     155.92     164.63     175.40     184.79     196.44     207.89
    220.00     233.87     246.38     262.51     277.18     294.00     311.83     329.26     350.81     369.58     392.88     415.77
    440.00     467.75     492.77     525.03     554.37     587.99     623.66     658.51     701.62     739.15     785.76     831.55
    880.00     935.49     985.54    1050.05    1108.73    1175.99    1247.32    1317.02    1403.24    1478.31    1571.53    1663.10
   1760.00    1870.98    1971.08    2100.11    2217.46    2351.97    2494.64    2634.04    2806.47    2956.61    3143.05    3326.19
   3520.00    3741.97    3942.15    4200.21    4434.92    4703.95    4989.29    5268.09    5612.95    5913.23    6286.11    6652.38
```

Fig. 2. Frequency values for all the tones of the piano tuned with Valloti's temperament.

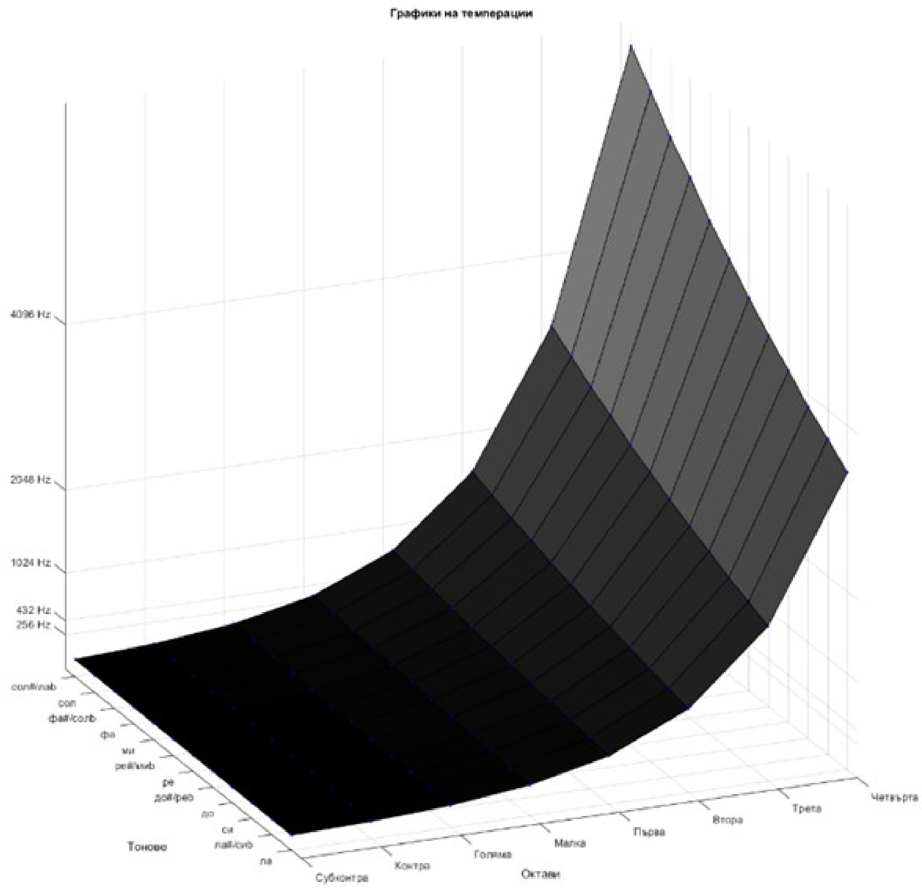


Fig. 3. Surface plot for all the tones of the piano tuned with Valloti's temperament.

A very useful option of the developed code is the possibility to overlap the surface plots of the temperaments (Fig. 3) so that it is obvious which tones in which octaves are higher or lower when compared. For example, there is the superposition of the Equal temeprament and the temeprament with open fifths (Fig. 4).

It can be observed that the equal temperament (more color abundant) is higher in the lower register and lower in the higher register than the Open Fifths Temperament (the blue-green gradient plot), except for the f-s and b-flat-s. This reflects

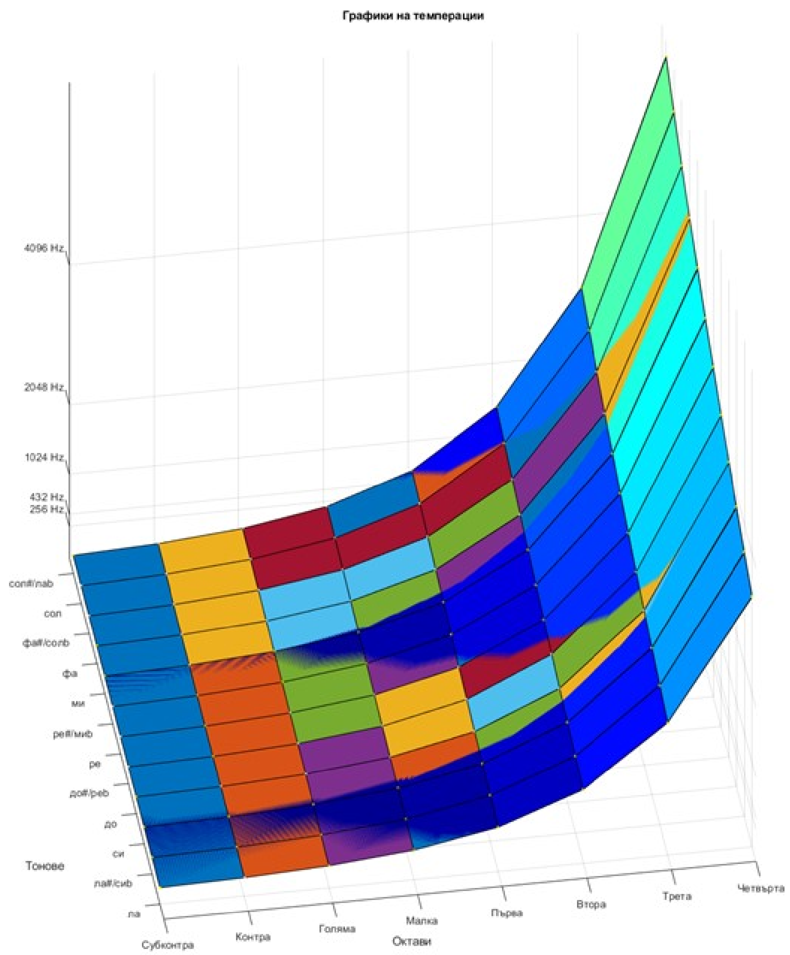
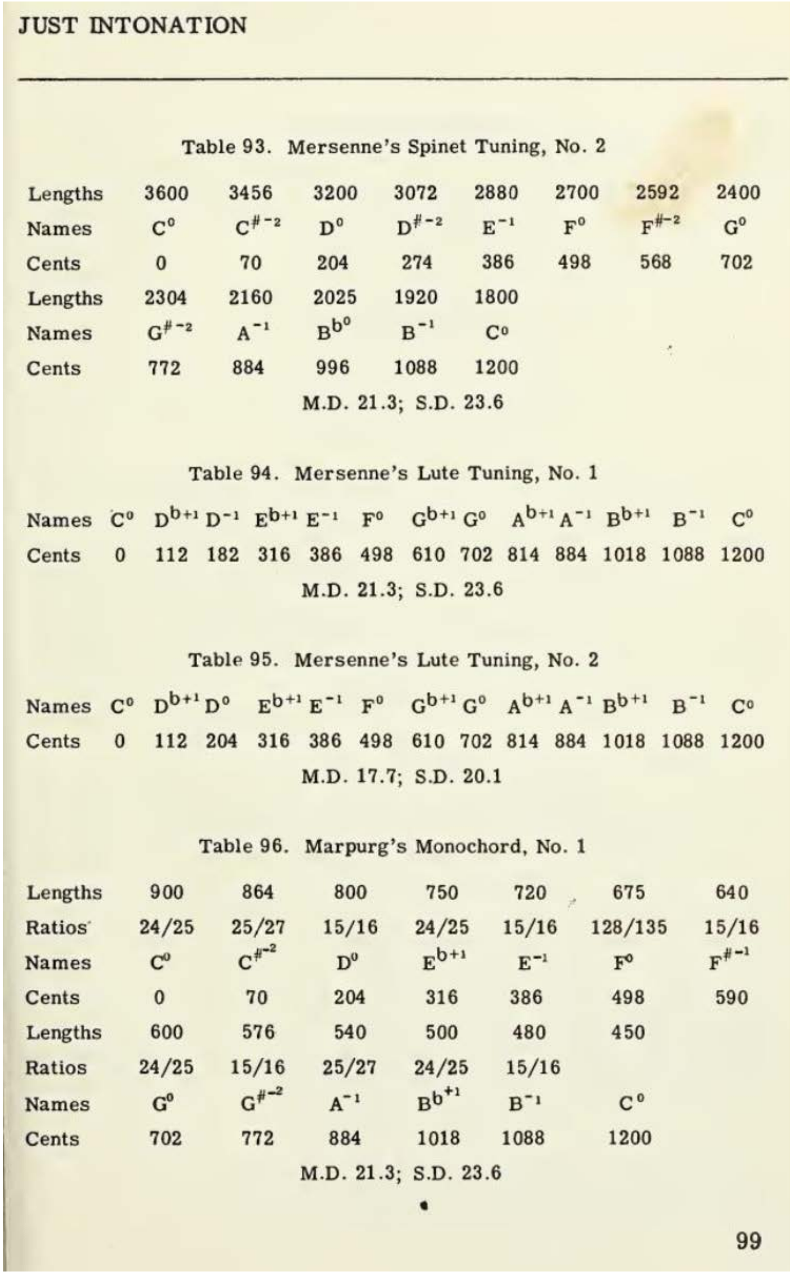


Fig. 4. Ovelapping surface plots for all the tones of the piano tuned with equal temperament and open fifths temperament.

the structure of the temperament with open fifths – it is an irregular noncirculatory temperament (opposite to the equal temperament, which is regular and circulatory).

3.2 Murray Barbours's tuning systems

Another useful place where modeling temperaments in MATLAB is helpful is for getting a deeper understanding of the tuning examples in J. Murrey Barbour's book "Tuning and Temperarnent. A Historical Survey" [8]. There Barbour gives a myriad of tuning examples in tables that contain lengths of stings, names of tones, ratios, and



JUST INTONATION

Table 93. Mersenne's Spinet Tuning, No. 2

Lengths	3600	3456	3200	3072	2880	2700	2592	2400
Names	C^0	$C^{\#-2}$	D^0	$D^{\#-2}$	E^{-1}	F^0	$F^{\#-2}$	G^0
Cents	0	70	204	274	386	498	568	702
Lengths	2304	2160	2025	1920	1800			
Names	$G^{\#-2}$	A^{-1}	B^{b^0}	B^{-1}	C^0			
Cents	772	884	996	1088	1200			

M.D. 21.3; S.D. 23.6

Table 94. Mersenne's Lute Tuning, No. 1

Names	C^0	D^{b+1}	D^{-1}	E^{b+1}	E^{-1}	F^0	G^{b+1}	G^0	A^{b+1}	A^{-1}	B^{b+1}	B^{-1}	C^0
Cents	0	112	182	316	386	498	610	702	814	884	1018	1088	1200

M.D. 21.3; S.D. 23.6

Table 95. Mersenne's Lute Tuning, No. 2

Names	C^0	D^{b+1}	D^0	E^{b+1}	E^{-1}	F^0	G^{b+1}	G^0	A^{b+1}	A^{-1}	B^{b+1}	B^{-1}	C^0
Cents	0	112	204	316	386	498	610	702	814	884	1018	1088	1200

M.D. 17.7; S.D. 20.1

Table 96. Marpurg's Monochord, No. 1

Lengths	900	864	800	750	720	675	640
Ratios	24/25	25/27	15/16	24/25	15/16	128/135	15/16
Names	C^0	$C^{\#-2}$	D^0	E^{b+1}	E^{-1}	F^0	$F^{\#-1}$
Cents	0	70	204	316	386	498	590
Lengths	600	576	540	500	480	450	
Ratios	24/25	15/16	25/27	24/25	15/16		
Names	G^0	$G^{\#-2}$	A^{-1}	B^{b+1}	B^{-1}	C^0	
Cents	702	772	884	1018	1088	1200	

M.D. 21.3; S.D. 23.6

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Fig. 5. A page of Barbour's book showing examples of four historical temperaments [8].

cents of the distance from the basic tone (Fig. 5).

For the purpose of modelling the temperament an OOP MATLAB class was developed that is capable of generating the frequencies of the tones of the tunings from any giver parameter – lengths, distance in cents, ratios (Fig. 6).

```
>> MST1 = temp

MST1 =

  temp with properties:

          num: []
        ratio: []
          len: 1000.00
         cent: []
         freq: 256.00
        steps: []
   stepsRatio: []
```

Fig. 6. Matlab's console output after the execution of the `temp.m` script.

One has to input the cents values into as a property of the class (Fig. 7).

```
>> MST1.cent = [0 70 204 274 386 498 568 702 772 884 996 1088 1200]

MST1 =

  temp with properties:

          num: []
        ratio: []
          len: 1000.00
         cent: [0 70.00 204.00 274.00 386.00 498.00 568.00 702.00 772.00 884.00 996.00 1088.00 1200.00]
         freq: 256.00
        steps: []
   stepsRatio: []
```

Fig. 7. Matlab's console output after entering the values of the width f the intervals, described by Barbour [8].

Then use the built in method to generate the whole set of properties from the input, which in this case are the cent values for the intervals (Fig. 8).

The class has methods for generating the whole set of frequencies for the full keyboard layout with the method `freqtable` and to generate a surface of the tuning with the `freqfigure` method.

The most important feature that the models in Section 3 have are the frequency tables. They are vital for being able to sonify the temperament.

4 TONAL SYSTEMS AND MUSICAL SPECTRUM

Another aspect of using MATLAB for modelling tonal systems is adapting tunings from timbre/spectrum and designing spectrum/timbre for tunings. This is done by using the proposed by William Setheres [9] concept for deriving timbre from the

```
>> MST1 = MST1.tempfromc

MST1 =

  temp with properties:

          num: [1.00 2.00 3.00 4.00 5.00 6.00 7.00 8.00 9.00 10.00 11.00 12.00 13.00]
        ratio: [1.00 1.04 1.13 1.17 1.25 1.33 1.39 1.50 1.56 1.67 1.78 1.87 2.00]
          len: [1000.00 960.37 888.84 853.62 800.14 750.02 720.30 666.65 640.23 600.12 562.53 533.42 500.00]
         cent: [0 70.00 204.00 274.00 386.00 498.00 568.00 702.00 772.00 884.00 996.00 1088.00 1200.00]
         freq: [256.00 266.56 288.01 299.90 319.94 341.32 355.41 384.01 399.86 426.58 455.09 479.93 512.00]
        steps: [70.00 134.00 70.00 112.00 112.00 70.00 134.00 70.00 112.00 112.00 92.00 112.00]
   stepsRatio: [1.04 1.08 1.04 1.07 1.07 1.04 1.08 1.04 1.07 1.07 1.05 1.07]
```

Fig. 8. Matlab's console output after generating the structure of the temperament from the input interval width using the `tempfromc` method.

structure of the scale. The concept is based on lowering the intrinsic dissonance that appear between the overtones in the spectrum of any complex tone and the structure of the scale. Tis is most useful for temperaments, especially for EDO systems. In the new spectrum, the structure of the overtones is altered in such a way that it reflects the mapping of the prime intervals, as shown in the val. The val represents the approximation to which how many numbers of stacked generators represent certain intervals.

The reversed process is done here – the width of the mapped intervals is used to create the overtones's structure.

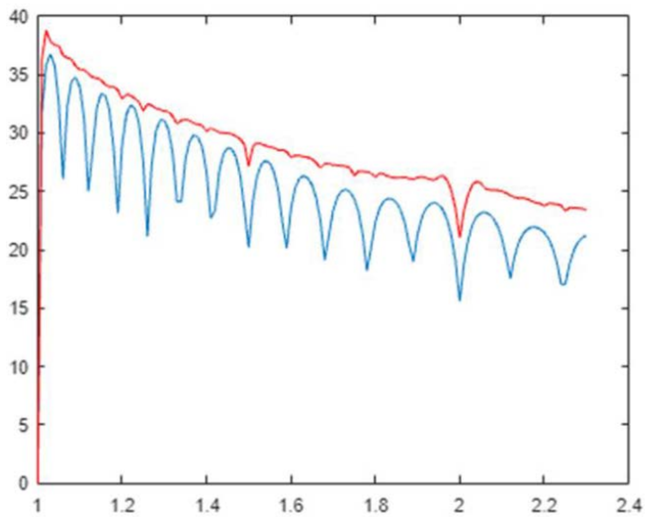

Fig. 9. Dissonance curve comparing the natural spectrum with 37 overtones (red line) and modified spectrum with 37 overtones (blue line) of the 12 EDO [10].

The image in Fig. 9 shows a dissonance curve of the overtones' series of a spectrum and its intrinsic dissonance with red, compared to the dissonance curve modified to accommodate the 12 EDO intervals instead of the rational intervals of the natural numbers. The curves are calculated by using 37 overtones. The modified dissonance curve shows closer correspondences (lower minima) to the width of the intervals of the 12 EDO, as the natural curve (red curve) has lower minima only at the octave and the pure fifth.

The overtone indices for the natural and the modified overtone series can be observed in the following figure:

```
>> ovrtnindx(1:2,:)

ans =

  Columns 1 through 8

        1.00        2.00        3.00        4.00        5.00        6.00        7.00        8.00
        1.00        2.00        3.00        4.00        5.04        5.99        7.13        8.00

  Columns 9 through 16

        9.00       10.00       11.00       12.00       13.00       14.00       15.00       16.00
        8.98       10.08       11.31       11.99       12.70       14.25       15.10       16.00

  Columns 17 through 24

       17.00       18.00       19.00       20.00       21.00       22.00       23.00       24.00
       16.95       17.96       19.03       20.16       21.36       22.63       22.63       23.97

  Columns 25 through 32

       25.00       26.00       27.00       28.00       29.00       30.00       31.00       32.00
       25.40       25.40       26.91       28.51       28.51       30.20       30.20       32.00

  Columns 33 through 37

       33.00       34.00       35.00       36.00       37.00
       33.90       33.90       35.92       35.92       38.05
```

Fig. 10. Table of overtone indices' values: first line - values of the natural overtones' indices; second line - adjusted values to the structure of the scale.

The table in Fig. 10 shows us that most of the overtones over the 4th are altered so that the spectrum can accommodate the intrinsic dissonance of the scale.

5 Generating WAV Files

All the proposed models of tuning systems are suitable for sonification by MATLAB's functions for generating wave files. The produced wave files can be processed into digital instrument using a sampler like Native instrument's KONTAKT player [11]. Further, these virtual instruments can be used in digital audio workstations [12] or music editing software.

6 Conclusion

Modeling tonal systems with Matlab is a fruitful initiative. It allows the researcher to develop deeper understanding of the structure of the model and the relations between the tones. One can build out of the mathematical model realistic audio representations and experiment with them in creative environments (digital audio workstations and composing software). Another useful aspect is incorporating the results into the musical education by providing reference of how different a musical piece may sound when the tonal system is changed. This can bring to musicians' understanding the reasoning behind some compositional practices that avoid certain intervals or note arrangements. Another aspect is providing musical performer an abundant pallet for accurately representing the historical style of music or presenting the composers with diverse options for incorporating novelty in the sound and at the same time leaning onto the tradition.

The application of Matlab can be extended far beyond the modelling of existing systems. Matlab is a good starting point for initiating research on the application of generative neural networks (for example Variational autoencoder [13]) in the musical domain.

Acknowledgements

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