Journal of Theoretical and Applied Mechanics, Sofia, Vol.55 (2025) pp. 234-249
SOLID MECHANICS

UNIQUENESS THEOREMS IN THE QUASI-STATIC PROBLEMS OF THE COUPLED LINEAR THEORY OF ELASTIC NANOMATERIALS WITH TRIPLE POROSITY

MARIAM MIKELASHVILI*

Ilia State University, K. Cholokashvili Ave., 3/5, 0179 Tbilisi, Georgia

[Received: 23 November 2024. Accepted: 23 May 2025]

doi: https://doi.org/10.55787/jtams.25.55.2.234

ABSTRACT: The paper focuses on the coupled linear quasi-static theory of elasticity for nanomaterials with triple porosity, incorporating concepts of Darcy's law and volume fractions of three levels of pores (macro-, meso-, and micropores). It formulates the general governing equations in terms of the displacement vector field, pores volume fractions and fluid pressures in the pore networks in three levels of pores. Then, the 3D internal and external basic boundary value problems (BVPs) of steady vibrations of this model are formulated. Green's first identity is obtained and finally, the uniqueness theorems for classical solutions of the above mentioned BVPs are proved.

KEY WORDS: Nanomaterials with triple porosity, quasi-static, uniqueness theorems.

1 INTRODUCTION

For more than a century, predicting the mechanical properties of materials with single and multiple porosity has been a key focus in continuum mechanics. The first quasi-static theory of poroelasticity, based on Darcy's law, was introduced by Biot in his paper [1], where he presented the coupling effect between fluid pressure and mechanical stress, also quasi-static mathematical model of elastic solids with double porosity was developed by Wilson and Aifantis [2]. Understanding the mechanical effects of coupled processes is essential for modern theories of porous media.

This paper discusses advancements in nanotechnology, particularly the development of nanomaterials with multiple porosities and their broad applications in various scientific fields. Nanomaterials, due to their unique multi-porous nanoscale structures, are widely used in medicine, biology, engineering, and technology. They find applications in areas such as medical diagnostics, food analysis, drug delivery, and decontamination.

Nanomaterials are typically categorized based on pore diameter size into three groups: micropores (with pore diameters less than 2 nm), mesopores (2-50 nm), and

*Corresponding author e-mail: marimikelashvili@gmail.com

macropores (with pore diameters greater than 50 nm). The mathematical models of triple porosity nanomaterials are based on quasi-static elasticity, involving the displacement vector field, fluid pressure changes at three pore levels, and changes in the volume fractions of these pore systems.

The mechanical properties of porous nanomaterials are naturally coupled, making it difficult to seperate one from the other. Coupled processes are essential for engineering, geological, and biological applications. The mechanical properties of porous nanomaterials are inherently coupled, making it challenging to isolate individual effects. However, coupled phenomena have been extensively studied in engineering, geological, and biological materials with single porosity. Comprehensive references on this topic can be found in works by Cheng [3], Wang [4], Selvadurai and Suvorov [5], and Stephanson [6] et al. Theoretical work on porous nanomaterials has evolved around two primary approaches: Darcy's law and the volume fraction concept. Biot's [1] pioneering work introduced a quasi-static mathematical model of poroelasticity based on Darcy's law, where fluid pressure and mechanical stress are coupled. This model has been widely applied to various engineering problems. Significant advancements in understanding these coupled processes have been achieved by numerous researchers. For further details, refer to studies by Booker and Savvidou [7], Derski and Kowalski [8], Kurashige [9], McTigue [10], Nguyen and Selvadurai [11], and Diebels [12]. An overview of foundational results in the theories of single and multi-porosity materials, particularly those grounded in Darcy's law, can be found in the books of Straughan [13] and Svanadze [14].

An alternative theory, introduced by Nunziato and Cowin [15], is based on the volume fraction concept, offering a more detailed description of the kinematic variables in porous media. Within this theory, the independent kinematic variables consist of the displacement vector field and the changes in pores volume fractions. More advanced models in elasticity and thermoelasticity for single-porosity materials, utilizing the volume fraction concept, have been introduced and extensively studied by multiple research groups. Foundational results in this field can be found in the works of Ciarletta and Ieşan [16], Ieşan [17], Straughan [13] [18] [19], and the references therein.The theory of Nunziato and Cowin [15], has been expanded to include mathematical models for deformable materials with double voids, developed by Ieşan and Quintanilla [20].

When a porous nanomaterial deforms, both the volume fraction of pores and the fluid pressure change simultaneously. As a result, new mathematical models must incorporate all three factors-the deformation of the skeleton, changes in the pore volume fraction, and fluid pressure within the pores. Extensive references on the coupled effects in porous media can be found in the works of Liu [21] and Stephanson et al. [22].

Recently, Svanadze [23], [24] introduced linear models for elastic and thermoelastic single-porosity materials, where the coupled interaction of Darcy's law and the volume fraction of pore networks is considered. The basic boundary value problems (BVPs) of steady vibrations in the quasi-static case of these models were studied by Bitsadze [25], [26] and Mikelashvili [27], [28]. Furthermore, Svanadze [29] investigated the steady vibration problems of viscoelastic single-porosity materials. More recently, the coupled phenomenon has been extended to double-porosity elastic and viscoelastic materials in the works of Svanadze [30], [31], where the basic BVPs were explored using the potential method. The (BVPs) for steady vibrations in the quasi-static case of these models have been explored by Mikelashvili [32], [33]. It is important to note that the potential method is essential for solving BVPs in mathematical physics and continuum mechanics. A comprehensive review of its historical development and bibliography can be found in the book by Kupradze et al. [34].

Recent work by Svanadze [35] has extended these models to triple porosity nanomaterials, combining Darcy's law and volume fraction effects for multiple pore levels. These models are important for applications in urban and geological engineering, mechanics, medicine, and biology.

In this paper, we propose a coupled linear quasi-static mathematical model of elasticity for nanomaterials with triple porosity, which integrates Darcy's law with the volume fraction concept for three levels of pores.

The paper is structured as follows: Section 2 presents the governing equations of quasi-static of motion and steady vibrations of the theory under consideration. Section 3 formulates the basic internal and external BVPs of steady vibrations. In Section 4, Green's first identity is established. Finally, in Section 5, on the basis of Green's identity the uniqueness theorems for classical solutions of the basic quasi-static BVPs are proved.

2 Governing Equations

Let $\mathbf{x} = (x_1, x_2, x_3)$ be a point of the Euclidean three-dimensional space $\mathbb{R}^3$ and let t denotes the time variable, $t \geq 0$. We assume that an isotropic and homogeneous elastic solid with triple porosity structure occupies a region of $\mathbb{R}^3$. This structure of nanomaterials means that the skeleton of solid consists of pores on the macro, meso and micro scales. Afterwards, in this section, functions and vectors that depend on the space variable $\mathbf{x}$ and the time t will be denoted with hat.

Let $\hat{\mathbf{u}} = (\hat{u}_1, \hat{u}_2, \hat{u}_3)$ be the displacement vector in solid, $\hat{\varphi}_l$ and $\hat{p}_l$ $(l = 1, 2, 3)$ are the changes of the volume fraction and the fluid pressure from the reference configuration for l-th pore network, respectively. Moreover, throughout this paper, we shall employ the usual summation and differentiation conventions: (i) repeated Latin indices are summed over the ranges (1, 2, 3); (ii) the subscripts preceded by a comma

denote partial differentiation with respect to the corresponding Cartesian coordinate; (iii) a superposed dot denotes differentiation with respect to t.

According to Svanadze [35], the governing system of field equations in the coupled linear quasi-static theory of elasticity for nanomaterials with triple porosity comprises the following four sets of equations:

• *Constitutive equations*

$$\begin{aligned} &\hat{t}_{lj} = 2\mu\,\hat{e}_{lj} + \lambda\,\hat{e}_{rr}\delta_{lj} + (b_r\,\hat{\varphi}_r - \beta_r\hat{p}_r)\delta_{lj}, \\ &\hat{\sigma}_j^{(l)} = a_{lr}\hat{\varphi}_{r,j}, \qquad l,j = 1,2,3. \end{aligned} \tag{1}$$

• *Equilibrium equations*

$$\hat{t}_{lj,j} = -\rho\hat{F}_l^{(1)}, \qquad \hat{\sigma}_{j,j}^{(l)} + \hat{\xi}^{(l)} = -\rho\hat{F}_l^{(2)}, \qquad l = 1,2,3. \tag{2}$$

• *Darcy's law for triple-porosity materials*

$$\hat{\mathbf{v}}^{(l)} = -\frac{k'_{lj}}{\mu'}\nabla\hat{p}_j - \hat{\mathbf{s}}^{(l)}. \tag{3}$$

• *Equations of fluid mass conservation*

$$\hat{v}_{j,j}^{(l)} + \dot{\hat{\zeta}}_l + \beta_l\dot{\hat{e}}_{jj} + \hat{d}_l = 0. \tag{4}$$

In these equations, $\hat{t}_{lj}$ is the component of total stress tensor, ρ is the reference mass density, $\rho > 0$, $\hat{\mathbf{F}}^{(1)} = (\hat{F}_1^{(1)}, \hat{F}_2^{(1)}, \hat{F}_3^{(1)})$, $\hat{\mathbf{F}}^{(2)} = (\hat{F}_1^{(2)}, \hat{F}_2^{(2)}, \hat{F}_3^{(2)})$ is the body force per unit mass, $\hat{\sigma}_j^{(l)}, \hat{\xi}^{(l)}$ and $\hat{\mathbf{s}}^{(l)} = (\hat{s}_1^{(l)}, \hat{s}_2^{(l)}, \hat{s}_3^{(l)})$ are the components of the equilibrated stress, the intrinsic equilibrated body force and the extrinsic equilibrated body force associated with l-th level of pores, $\hat{e}_{lj}$ is the component of strain tensor and given by

$$\hat{\xi}^{(l)} = -b_l\hat{e}_{rr} - c_{lj}\hat{\varphi}_j + m_{lj}\hat{p}_j, \qquad l = 1,2,3. \tag{5}$$

$$\hat{e}_{lj} = \frac{1}{2}(\hat{u}_{l,j} + \hat{u}_{j,l}) \tag{6}$$

and

$$\hat{d}_l = \sum_{j=1,j\neq l}^{3} \gamma_{l+j-2}(\hat{p}_l - \hat{p}_j),$$

λ and μ are the Lamé constants, β_l are the effective stress parameters, δ_{lj} is Kronecker's delta, $\hat{\mathbf{v}}^{(l)} = (\hat{v}_1^{(l)}, \hat{v}_2^{(l)}, \hat{v}_3^{(l)})$ are the fluid flux vector for the l-th pore network, γ_l is the internal transport coefficient (leakage parameter) corresponding to a

fluid transfer rate respecting the intensity of flow between the pore phases, $\gamma_l \geq 0$, $\hat{\zeta}_l$ is the increment of fluid (volumetric strain) in the l-th pore phase and defined by

$$\hat{\zeta}_l = \alpha_{lj}\hat{p}_j + m_{lj}\hat{\varphi}_j, \tag{7}$$

α_{lj} is the cross-coupling compressibility for a fluid flow at the interface between the three pore systems and measures the compressibility of pore systems, δ_{lj} is the Kronecker delta, m_{lj} is the coefficient of cross-correlation term $m_{lj} = m_{jl}$, ∇ is the gradient operator, b_l, c_{lj}, α_{lj} and k'_{lj} $(l, j = 1, 2, 3)$ are the constitutive coefficients.

Substituting Eqs. (1), (3) and (5)-(7) into (2) and (4) we obtain the following system of equations in the coupled linear quasi-static theory of elastic triple-porosity nanomaterials expressed in terms of the displacement vector $\hat{\mathbf{u}}$, the changes of the volume fractions $\hat{\boldsymbol{\varphi}} = (\hat{\varphi}_1, \hat{\varphi}_2, \hat{\varphi}_3)$ and the changes of the fluid pressures $\hat{\mathbf{p}} = (\hat{p}_1, \hat{p}_2, \hat{p}_3)$:

$$\begin{gathered}
\mu\Delta\hat{u}_l + (\lambda+\mu)\,\hat{u}_{j,lj} + b_j\hat{\varphi}_{j,l} - \beta_j\hat{p}_{j,l} = -\rho\hat{F}^{(1)}_l, \\
(a_{lj}\Delta - c_{lj})\hat{\varphi}_j - b_l\,\hat{u}_{j,j} + m_{lj}\hat{p}_j = -\rho\hat{F}^{(2)}_l, \\
k_{lj}\Delta\hat{p} - \alpha_{lj}\hat{p} - d_{lj}\hat{p}_j - \beta_l\,\dot{\hat{u}}_{j,j} - m_{lj}\dot{\hat{\varphi}}_j = \hat{\mathbf{s}}^{(l)}_{j,j},
\end{gathered} \tag{8}$$

where Δ is the Laplacian operator and

$$k_{lj} = \frac{\kappa_{lj}}{\mu'}, \qquad d_{11} = \gamma_1 + \gamma_2, \qquad d_{22} = \gamma_1 + \gamma_3, \qquad d_{33} = \gamma_2 + \gamma_3,$$
$$d_{12} = d_{21} = -\gamma_1, \qquad d_{13} = d_{31} = -\gamma_2, \qquad d_{23} = d_{32} = -\gamma_3.$$

If we assume that $\hat{u}_l, \hat{F}^{(1)}_l, \hat{F}^{(2)}_l, \hat{\varphi}_l, \hat{p}_l, \hat{\mathbf{s}}^{(l)}$ $(l = 1, 2, 3)$ are postulated to have a harmonic time variation

$$\left\{\hat{u}_l, \hat{F}^{(1)}_l, \hat{F}^{(2)}_l, \hat{\varphi}_l, \hat{p}_l, \hat{\mathbf{s}}^{(l)}\right\}(\mathbf{x}, t) = \mathrm{Re}\left[\left\{u_l, F^{(1)}_l, F^{(2)}_l, \varphi_l, p_l, \mathbf{s}^{(l)}\right\}(\mathbf{x})\, e^{-i\omega t}\right],$$

then from (8) we obtain the following system of equations of steady vibrations in the considered theory:

$$\begin{gathered}
\mu\Delta u_l + (\lambda+\mu)\,u_{j,lj} + b_j\varphi_{j,l} - \beta_j p_{j,l} = -\rho F^{(1)}_l, \\
(a_{lj}\Delta - c_{lj})\varphi_j - b_l\,u_{j,j} + m_{lj}p_j = -\rho F^{(2)}_l, \\
(k_{lj}\Delta + \alpha'_{lj})p_j + \beta'_l\,u_{j,j} + m'_{lj}\varphi_j = -\mathrm{div}\mathbf{s}^{(l)},
\end{gathered} \tag{9}$$

where $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{F}^{(1)}_l = (F^{(1)}_1, F^{(1)}_2, F^{(1)}_3)$, $\mathbf{F}^{(2)}_l = (F^{(2)}_1, F^{(2)}_2, F^{(2)}_3)$, ω is the oscillation frequency, $\omega > 0$, $\beta'_l = i\omega\beta_l$, $\alpha'_{lj} = i\omega\alpha_{lj} - d_{lj}$, $m'_{lj} = i\omega m_{lj}$, $\gamma'_r = i\omega\gamma_r - \gamma_0$, $\gamma'_3 = i\omega\gamma_3 + \gamma_0$ $(l, j = 1, 2, 3, r = 1, 2)$.

For subsequent analysis, we require the following second-order matrix differential operator with constant coefficients:

$$\mathbf{M}(\mathbf{D_x}) = (M_{lj}(\mathbf{D_x}))_{9\times 9}\,, \qquad M_{lj} = \mu\Delta\delta_{lj} + (\lambda+\mu)\frac{\partial^2}{\partial x_l \partial x_j},$$

$$M_{l;j+3} = -M_{l+3;j} = b_j\frac{\partial}{\partial x_l}, \qquad M_{l;j+6} = -\beta_j\frac{\partial}{\partial x_l}, \qquad M_{l+3;j+3} = a_{lj}\Delta - c_{lj},$$

$$M_{l+3;j+6} = m_{lj}, \qquad M_{l+6;j} = \beta'_l\frac{\partial}{\partial x_j}, \qquad M_{l+6;j+3} = m'_{lj},$$

$$M_{l+6;j+6} = k_{lj}\Delta + \alpha'_{lj}, \qquad \mathbf{D_x} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right), \qquad l,j = 1,2,3.$$

It is straightforward to see that system (9) can be reformulated as follows:

$$\mathbf{M}(\mathbf{D_x})\,\mathbf{U}(\mathbf{x}) = \mathbf{F}(\mathbf{x}), \tag{10}$$

where $\mathbf{U} = (\mathbf{u}, \boldsymbol{\varphi}, \mathbf{p})$ and $\mathbf{F} = (-\rho\mathbf{F}^{(1)}, -\rho\mathbf{F}^{(2)}, -\mathrm{div}\hat{\mathbf{s}}^{(1)}, -\mathrm{div}\hat{\mathbf{s}}^{(2)}, -\mathrm{div}\hat{\mathbf{s}}^{(3)})$ are nine-component vector functions, $\mathbf{x}\in\mathbb{R}^3$.

3 Boundary Value Problems

Let S be the closed surface surrounding the finite domain Ω^+ in $\mathbb{R}^3$, $S\in C^{1,\nu}$, $0<\nu\le 1$, $\overline{\Omega^+} = \Omega^+\cup S$, $\Omega^- = \mathbb{R}^3\backslash\overline{\Omega^+}$, $\overline{\Omega^-} = \Omega^-\cup S$; $\mathbf{n}(\mathbf{z})$ is the external (with respect to Ω^+) unit normal vector to S at $\mathbf{z}$.

In what follows, the scalar product of two vectors $\mathbf{U} = (U_1, U_2, \cdots, U_l)$ and $\mathbf{U}' = (U'_1, U'_2, \cdots, U'_l)$ is denoted by $\mathbf{U}\cdot\mathbf{U}' = \sum\limits_{j=1}^{l} U_j\overline{U'_j}$, where $\overline{U'_j}$ is the complex conjugate of U'_j.

Definition. A vector function $\mathbf{U} = (U_1, U_2, \cdots, U_9)$ is called *regular* in Ω^- (or Ω^+) if

(i)

$$U_l \in C^2(\Omega^-)\cap C^1(\overline{\Omega^-}) \qquad (or\ \ U_l\in C^2(\Omega^+)\cap C^1(\overline{\Omega^+})),$$

(ii)

$$U_l(\mathbf{x}) = O(|\mathbf{x}|^{-1}), \qquad U_{l,j}(\mathbf{x}) = o(|\mathbf{x}|^{-1}), \tag{11}$$

for $|\mathbf{x}|\gg 1$, where $l = 1,2,\cdots,9$ and $j = 1,2,3$.

In the sequel, we use the matrix differential operator

$$\mathbf{R}(\mathbf{D_x}, \mathbf{n}) = (R_{lj}(\mathbf{D_x}, \mathbf{n}))_{9\times 9},$$

where

$$R_{lj}(\mathbf{D_x},\mathbf{n})=\mu\delta_{lj}\frac{\partial}{\partial\mathbf{n}}+\mu n_j\frac{\partial}{\partial x_l}+\lambda n_l\frac{\partial}{\partial x_j},\qquad R_{lr}(\mathbf{D_x},\mathbf{n})=b_{r-3}\,n_l,$$

$$R_{l;r+3}(\mathbf{D_x},\mathbf{n})=-\beta_{r-3}\,n_l,\qquad R_{l+3;j+3}(\mathbf{D_x},\mathbf{n})=a_{lj}\frac{\partial}{\partial\mathbf{n}},$$

$$R_{l+3;j}(\mathbf{D_x},\mathbf{n})=R_{l+3;j+6}(\mathbf{D_x},\mathbf{n})=R_{l+6;j}(\mathbf{D_x},\mathbf{n})=R_{l+6;j+3}(\mathbf{D_x},\mathbf{n})=0,$$

$$R_{l+6;j+6}(\mathbf{D_x},\mathbf{n})=k_{lj}\frac{\partial}{\partial\mathbf{n}},\quad l,j=1,2,3,\quad r=4,5,6$$

and $\dfrac{\partial}{\partial\mathbf{n}}$ is the derivative along the vector $\mathbf{n}$.

The basic internal and external BVPs of steady vibrations in the coupled linear quasi-static theory of elasticity for nanomaterials with triple porosity are formulated as follows.

Find a regular (classical) solution to system (10) for $\mathbf{x}\in\Omega^+$ satisfying the boundary condition

$$\lim_{\Omega^+\ni\mathbf{x}\to\mathbf{z}\in S}\mathbf{U}(\mathbf{x})\equiv\{\mathbf{U}(\mathbf{z})\}^+=\mathbf{f}(\mathbf{z})$$

in the internal *Problem* $(I)^+_{\mathbf{F},\mathbf{f}}$, and

$$\lim_{\Omega^+\ni\mathbf{x}\to\mathbf{z}\in S}\mathbf{R}(\mathbf{D_x},\mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{x})\equiv\{\mathbf{R}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{z})\}^+=\mathbf{f}(\mathbf{z})$$

in the internal *Problem* $(II)^+_{\mathbf{F},\mathbf{f}}$, where $\mathbf{F}$ and $\mathbf{f}$ are prescribed nine-component vector functions.

Find a regular (classical) solution to system (10) for $\mathbf{x}\in\Omega^-$ satisfying the boundary condition

$$\lim_{\Omega^-\ni\mathbf{x}\to\mathbf{z}\in S}\mathbf{U}(\mathbf{x})\equiv\{\mathbf{U}(\mathbf{z})\}^-=\mathbf{f}(\mathbf{z})$$

in the external *Problem* $(I)^-_{\mathbf{F},\mathbf{f}}$, and

$$\lim_{\Omega^-\ni\mathbf{x}\to\mathbf{z}\in S}\mathbf{R}(\mathbf{D_x},\mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{x})\equiv\{\mathbf{R}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{z})\}^-=\mathbf{f}(\mathbf{z})$$

in the external *Problem* $(II)^-_{\mathbf{F},\mathbf{f}}$, where $\mathbf{F}$ and $\mathbf{f}$ are prescribed nine-component vector functions.

Our goal is to prove the uniqueness theorems of classical solutions of the BVPs $(I)^\pm_{\mathbf{F},\mathbf{f}}$ and $(II)^\pm_{\mathbf{F},\mathbf{f}}$, where we need Green's first identity.

4 GREEN'S IDENTITY

In this section, we derive Green's first identity in the coupled linear quasi-static theory of elasticity for nanomaterials with triple porosity. Subsequently, we prove uniqueness theorems for the regular (classical) solutions of the BVPs $(I)^{\pm}_{\mathbf{F},\mathbf{f}}$ and $(II)^{\pm}_{\mathbf{F},\mathbf{f}}$.

In the sequel we use the matrix differential operators:

$$1)\quad \begin{aligned} &\mathbf{M}^{(0)}(\mathbf{D_x}) = \Big(M^{(0)}_{lj}(\mathbf{D_x})\Big)_{3\times 3}, \quad M^{(0)}_{lj}(\mathbf{D_x}) = \mu\Delta\delta_{lj} + (\lambda+\mu)\frac{\partial^2}{\partial x_l \partial x_j},\\ &\mathbf{M}^{(j)}(\mathbf{D_x}) = \Big(M^{(j)}_{lm}(\mathbf{D_x})\Big)_{3\times 9}, \quad M^{(j)}_{lm}(\mathbf{D_x}) = M_{l+3(j-1);m}(\mathbf{D_x}), \end{aligned}$$

$$2)\quad \begin{aligned} &\mathbf{R}^{(0)}(\mathbf{D_x},\mathbf{n}) = \Big(R^{(0)}_{lj}(\mathbf{D_x},\mathbf{n})\Big)_{3\times 3}, \qquad R^{(0)}_{lj}(\mathbf{D_x},\mathbf{n}) = R_{lj}(\mathbf{D_x},\mathbf{n}),\\ &\mathbf{R}^{(1)}(\mathbf{D_x},\mathbf{n}) = \Big(R^{(1)}_{lm}(\mathbf{D_x},\mathbf{n})\Big)_{3\times 9}, \qquad R^{(1)}_{lm}(\mathbf{D_x},\mathbf{n}) = R_{lm}(\mathbf{D_x},\mathbf{n}),\\ &\mathbf{R}^{(2)}(\mathbf{D_x},\mathbf{n}) = \Big(R^{(2)}_{lj}(\mathbf{D_x},\mathbf{n})\Big)_{3\times 3}, \qquad R^{(2)}_{lj}(\mathbf{D_x},\mathbf{n}) = a_{lj}\frac{\partial}{\partial \mathbf{n}},\\ &\mathbf{R}^{(3)}(\mathbf{D_x},\mathbf{n}) = \Big(R^{(3)}_{lj}(\mathbf{D_x},\mathbf{n})\Big)_{3\times 3}, \qquad R^{(3)}_{lj}(\mathbf{D_x},\mathbf{n}) = k_{lj}\frac{\partial}{\partial \mathbf{n}}, \end{aligned}$$

where $l,j = 1,2,3$ and $m = 1,2,\cdots,9$.

Let $\mathbf{u}' = (u'_1,u'_2,u'_3)$, $\boldsymbol{\varphi}' = (\varphi'_1,\varphi'_2,\varphi'_3)$ and $\mathbf{p}' = (p'_1,p'_2,p'_3)$ be three-component vector functions, $\mathbf{U}' = (\mathbf{u}',\boldsymbol{\varphi}',\mathbf{p}')$.

In what follows, we use the following functions:

$$(12)\quad \begin{aligned} W^{(0)}(\mathbf{u},\mathbf{u}') &= \frac{1}{3}(3\lambda+2\mu)\,\mathrm{div}\mathbf{u}\,\mathrm{div}\overline{\mathbf{u}'}\\ &\quad + \frac{\mu}{2}\sum_{l,j=1;l\neq j}^{3}(u_{l,j}+u_{j,l})(\overline{u'_{l,j}}+\overline{u'_{j,l}})\\ &\quad + \frac{\mu}{3}\sum_{l,j=1}^{3}\left(\frac{\partial u_l}{\partial x_l}-\frac{\partial u_j}{\partial x_j}\right)\left(\frac{\partial \overline{u'_l}}{\partial x_l}-\frac{\partial \overline{u'_j}}{\partial x_j}\right),\\ W^{(1)}(\mathbf{U},\mathbf{u}') &= W^{(0)}(\mathbf{u},\mathbf{u}') + (b_l\,\varphi_l - \beta_l p_l)\,\mathrm{div}\overline{\mathbf{u}'},\\ W^{(2)}(\mathbf{U},\boldsymbol{\varphi}') &= a_{lj}\nabla\varphi_j\cdot\nabla\overline{\varphi'_l} + (b_l\,\mathrm{div}\mathbf{u} + c_{lj}\varphi_j - m_{lj}p_j)\overline{\varphi'_l},\\ W^{(3)}(\mathbf{U},\mathbf{p}') &= k_{lj}\nabla p_j\cdot\nabla\overline{p'_l} - (\beta'_l\,\mathrm{div}\mathbf{u} + m'_{lj}\varphi_j + \alpha'_{lj}p_j)\overline{p'_l}. \end{aligned}$$

The following Lemmas will be useful to study the uniqueness of classical solutions to the BVPs.

Lemma 1. *If* $\mathbf{U} = (\mathbf{u}, \boldsymbol{\varphi}, \mathbf{p})$ *is a regular vector in* Ω^+, $u'_j, \varphi'_j, p'_j \in C^1(\Omega^+) \cap C(\overline{\Omega^+})$, $j = 1, 2, 3$, *then*

$$
\begin{aligned}
&\int\limits_{\Omega^+} \left[\mathbf{M}^{(1)}(\mathbf{D}_\mathbf{x})\,\mathbf{U}\cdot\mathbf{u}' + W^{(1)}(\mathbf{U},\mathbf{u}')\right] d\mathbf{x} = \int\limits_S \mathbf{R}^{(1)}(\mathbf{D}_\mathbf{z},\mathbf{n})\mathbf{U}\cdot\mathbf{u}'\, d_\mathbf{z}S,\\
&\int\limits_{\Omega^+} \left[\mathbf{M}^{(2)}(\mathbf{D}_\mathbf{x})\,\mathbf{U}\cdot\boldsymbol{\varphi}' + W^{(2)}(\mathbf{U},\boldsymbol{\varphi}')\right] d\mathbf{x} = \int\limits_S \mathbf{R}^{(2)}(\mathbf{D}_\mathbf{z},\mathbf{n})\boldsymbol{\varphi}\cdot\boldsymbol{\varphi}'\, d_\mathbf{z}S,\\
&\int\limits_{\Omega^+} \left[\mathbf{M}^{(3)}(\mathbf{D}_\mathbf{x})\,\mathbf{U}\cdot\mathbf{p}' + W^{(3)}(\mathbf{U},\mathbf{p}')\right] d\mathbf{x} = \int\limits_S \mathbf{R}^{(3)}(\mathbf{D}_\mathbf{z},\mathbf{n})\mathbf{p}\cdot\mathbf{p}'\, d_\mathbf{z}S.
\end{aligned}
\tag{13}
$$

Proof. On the basis of Green's first identity of the classical theory of elasticity (see, e.g., Kupradze et al. [36])

$$
\int\limits_{\Omega^+} \left[\mathbf{M}^{(0)}(\mathbf{D}_\mathbf{x})\,\mathbf{u}(\mathbf{x})\cdot\mathbf{u}'(\mathbf{x}) + W^{(0)}(\mathbf{u},\mathbf{u}')\right] d\mathbf{x} = \int\limits_S \mathbf{R}^{(0)}(\mathbf{D}_\mathbf{z},\mathbf{n})\mathbf{u}(\mathbf{z})\cdot\mathbf{u}'(\mathbf{z})\, d_\mathbf{z}S
$$

and with the next well-known formula

$$
\int\limits_{\Omega^+} \left[\nabla\varphi_l\cdot\mathbf{u}' + \varphi_l \mathrm{div}\overline{\mathbf{u}'}\right] d\mathbf{x} = \int\limits_S \varphi_l\mathbf{n}\cdot\mathbf{u}'\, d_\mathbf{z}S,
$$

we obtain the first relation of (13).

On the other hand, the divergence theorem leads to the following famous identity

$$
\int\limits_{\Omega^+} \left[\Delta\varphi_l(\mathbf{x})\,\overline{\varphi'_j(\mathbf{x})} + \nabla\varphi_l(\mathbf{x})\cdot\nabla\varphi'_j(\mathbf{x})\right] d\mathbf{x} = \int\limits_S \frac{\partial\varphi_l(\mathbf{z})}{\partial\mathbf{n}(\mathbf{z})}\,\overline{\varphi'_j(\mathbf{z})}\, d_\mathbf{z}S. \tag{14}
$$

Now, in view of the relations (12), from (14), we may derive the last two relations of (13). □

Lemma 1 and the condition at infinity (13) lead the following result.

Lemma 2. *If* $\mathbf{U} = (\mathbf{u}, \boldsymbol{\varphi}, \mathbf{p})$ *and* $\mathbf{U}' = (\mathbf{u}', \boldsymbol{\varphi}', \mathbf{p}')$ *are regular vectors in* Ω^-, *then*

$$
\begin{aligned}
&\int\limits_{\Omega^-} \left[\mathbf{M}^{(1)}(\mathbf{D}_\mathbf{x})\,\mathbf{U}\cdot\mathbf{u}' + W^{(1)}(\mathbf{U},\mathbf{u}')\right] d\mathbf{x} = -\int\limits_S \mathbf{R}^{(1)}(\mathbf{D}_\mathbf{z},\mathbf{n})\mathbf{U}\cdot\mathbf{u}'\, d_\mathbf{z}S,\\
&\int\limits_{\Omega^-} \left[\mathbf{M}^{(2)}(\mathbf{D}_\mathbf{x})\,\mathbf{U}\cdot\boldsymbol{\varphi}' + W^{(2)}(\mathbf{U},\boldsymbol{\varphi}')\right] d\mathbf{x} = -\int\limits_S \mathbf{R}^{(2)}(\mathbf{D}_\mathbf{z},\mathbf{n})\boldsymbol{\varphi}\cdot\boldsymbol{\varphi}'\, d_\mathbf{z}S,\\
&\int\limits_{\Omega^-} \left[\mathbf{M}^{(3)}(\mathbf{D}_\mathbf{x})\,\mathbf{U}\cdot\mathbf{p}' + W^{(3)}(\mathbf{U},\mathbf{p}')\right] d\mathbf{x} = -\int\limits_S \mathbf{R}^{(3)}(\mathbf{D}_\mathbf{z},\mathbf{n})\mathbf{p}\cdot\mathbf{p}'\, d_\mathbf{z}S.
\end{aligned}
\tag{15}
$$

The subsequent results follow from the Lemmas 1 and 2.

Theorem 1. *If* $\mathbf{U}=(\mathbf{u},\boldsymbol{\varphi},\mathbf{p})$ *and* $\mathbf{U}'=(\mathbf{u}',\boldsymbol{\varphi}',\mathbf{p}')$ *are regular vectors in* Ω^+*, then*

$$\int_{\Omega^+}\left[\mathbf{M}(\mathbf{D_x})\mathbf{U}(\mathbf{x})\cdot\mathbf{U}'(\mathbf{x})+W(\mathbf{U},\mathbf{U}')\right]d\mathbf{x}=\int_S\mathbf{R}(\mathbf{D_z},\mathbf{n})\mathbf{U}(\mathbf{z})\cdot\mathbf{U}'(\mathbf{z})\,d_{\mathbf{z}}S, \tag{16}$$

where

$$W(\mathbf{U},\mathbf{U}')=W^{(1)}(\mathbf{U},\mathbf{u}')+W^{(2)}(\mathbf{U},\boldsymbol{\varphi}')+W^{(3)}(\mathbf{U},\mathbf{p}').$$

Theorem 2. *If* $\mathbf{U}=(\mathbf{u},\boldsymbol{\varphi},\mathbf{p})$ *and* $\mathbf{U}'=(\mathbf{u}',\boldsymbol{\varphi}',\mathbf{p}')$ *are regular vectors in* Ω^-*, then*

$$\int_{\Omega^-}\left[\mathbf{M}(\mathbf{D_x})\,\mathbf{U}(\mathbf{x})\cdot\mathbf{U}'(\mathbf{x})+W(\mathbf{U},\mathbf{U}')\right]d\mathbf{x}=-\int_S\mathbf{R}(\mathbf{D_z},\mathbf{n})\mathbf{U}(\mathbf{z})\cdot\mathbf{U}'(\mathbf{z})\,d_{\mathbf{z}}S. \tag{17}$$

Formulas (16) and (17) represent Green's first identity in the coupled linear quasi-static theory of elasticity of triple-porosity nanomaterials for the domains Ω^+ and Ω^-, respectively.

5 Uniqueness Theorems

In this section, the uniqueness theorems for the regular (classical) solutions of the BVPs $(I)^{\pm}_{\mathbf{F},\mathbf{f}}$ and $(II)^{\pm}_{\mathbf{F},\mathbf{f}}$ are proved. It is easy to verify that from (12) we get

$$\begin{aligned}
&W^{(0)}(\mathbf{u},\mathbf{u})=\frac{1}{3}(3\lambda+2\mu)\,|\mathrm{div}\mathbf{u}|^2+W_0(\mathbf{u}),\\
&W^{(1)}(\mathbf{U},\mathbf{u})=W^{(0)}(\mathbf{u},\mathbf{u})+(b_l\,\varphi_l-\beta_l p_l)\,\mathrm{div}\overline{\mathbf{u}},\\
&W^{(2)}(\mathbf{U},\boldsymbol{\varphi})=a_{lj}\nabla\varphi_j\cdot\nabla\varphi_l'+(b_l\,\mathrm{div}\mathbf{u}+c_{lj}\varphi_j-m_{lj}p_j)\overline{\varphi_l},\\
&W^{(3)}(\mathbf{U},\mathbf{p})=k_{lj}\nabla p_j\cdot\nabla p_l'-(\beta_l'\,\mathrm{div}\mathbf{u}+m_{lj}'\varphi_j+\alpha_{lj}'p_j)\overline{p_l'},
\end{aligned} \tag{18}$$

where

$$W_0(\mathbf{u})=\frac{\mu}{2}\sum_{l,j=1;\,l\neq j}^{3}\left|\frac{\partial u_j}{\partial x_l}+\frac{\partial u_l}{\partial x_j}\right|^2+\frac{\mu}{3}\sum_{l,j=1}^{3}\left|\frac{\partial u_l}{\partial x_l}-\frac{\partial u_j}{\partial x_j}\right|^2. \tag{19}$$

We are now in a position to investigate the uniqueness of regular solutions of the BVPs $(I)^{\pm}_{\mathbf{F},\mathbf{f}}$ and $(II)^{\pm}_{\mathbf{F},\mathbf{f}}$. We have the following results.

In the following theorems, we will apply the matrix $\zeta=(\zeta_{lj})_{4\times4}$, where

$$\zeta_{lj}=c_{lj},\qquad \zeta_{4l}=\zeta_{l4}=b_l,\qquad \zeta_{44}=\frac{1}{3}(3\lambda+2\mu).$$

The following uniqueness theorems are valid.

Theorem 3. *If $\mu > 0$, $(\zeta_{lj})_{4\times4}$ and $(k_{lj})_{3\times3}$ are the positive definite matrices, $(a_{lj})_{3\times3}$ and $(\alpha_{lj})_{3\times3}$ are the non-negative definite matrices, then the internal BVP $(I)^{+}_{\mathbf{F},\mathbf{f}}$ admits at most one regular solution.*

Proof. Suppose that there are two regular solutions of problem $(I)^{+}_{\mathbf{F},\mathbf{f}}$. Then their difference $\mathbf{U}$ is a regular solution of the internal homogeneous BVP $(I)^{+}_{\mathbf{0},\mathbf{0}}$. Hence, $\mathbf{U}$ is a regular solution of the homogeneous equation

$$\mathbf{M}(\mathbf{D}_{\mathbf{x}})\mathbf{U}(\mathbf{x}) = \mathbf{0} \tag{20}$$

for $\mathbf{x} \in \Omega^{+}$, satisfying the homogeneous boundary condition

$$\{\mathbf{U}(\mathbf{z})\}^{+} = \mathbf{0} \qquad \text{for} \quad \mathbf{z} \in S. \tag{21}$$

Clearly, by virtue of (20) and (21), from (13) it follows that

$$\int_{\Omega^{+}} W^{(1)}(\mathbf{U},\mathbf{u})d\mathbf{x} = 0, \quad \int_{\Omega^{+}} W^{(2)}(\mathbf{U},\boldsymbol{\varphi})d\mathbf{x} = 0, \quad \int_{\Omega^{+}} W^{(3)}(\mathbf{U},\mathbf{p})d\mathbf{x} = 0. \tag{22}$$

In view of the relations (18) we can easily verify that

$$\begin{aligned}
&\mathrm{Re}W^{(1)}(\mathbf{U},\mathbf{u}) = W_0(\mathbf{u}) + b_l\mathrm{Re}(\varphi_l\mathrm{div}\overline{\mathbf{u}}) - \beta_l\mathrm{Re}(p_l\mathrm{div}\overline{\mathbf{u}}),\\
&\mathrm{Re}W^{(2)}(\mathbf{U},\boldsymbol{\varphi}) = a_{lj}\nabla\varphi_j\cdot\nabla\varphi_l + b_l\mathrm{Re}(\varphi_l\mathrm{div}\overline{\mathbf{u}}) + c_{lj}\varphi_j\overline{\varphi_l} - m_{lj}\mathrm{Re}(\varphi_l\overline{p_j}),\\
&\mathrm{Im}W^{(3)}(\mathbf{U},\mathbf{p}) = -\omega\beta_l\mathrm{Re}(p_l\mathrm{div}\overline{\mathbf{u}}) - \omega m_{lj}\mathrm{Re}(\overline{p_l}\varphi_j) - \omega\alpha_{lj}p_j\overline{p_l}
\end{aligned}$$

and consequently, we can write

$$\begin{aligned}
&\mathrm{Re}W^{(1)}(\mathbf{U},\mathbf{u}) + \mathrm{Re}W^{(2)}(\mathbf{U},\boldsymbol{\varphi}) - \frac{1}{\omega}\mathrm{Im}W^{(3)}(\mathbf{U},\mathbf{p})\\
&\quad = W_0(\mathbf{u}) + \left[\frac{1}{3}(3\lambda+2\mu)|\mathrm{div}\mathbf{u}|^2 + 2b_l\mathrm{Re}(\varphi_l\mathrm{div}\overline{\mathbf{u}}) + c_{lj}\varphi_j\overline{\varphi_l}\right]\\
&\qquad + a_{lj}\nabla\varphi_j\cdot\nabla\varphi_l + \alpha_{lj}p_j\overline{p_l}.
\end{aligned} \tag{23}$$

Based on the conditions on the constitutive coefficients of this Theorem and relation (19), from equation (23) we obtain

$$\begin{aligned}
&W_0(\mathbf{u}) \geq 0, \qquad a_{lj}\nabla\varphi_j\cdot\nabla\varphi_l \geq 0, \qquad \alpha_{lj}p_j\overline{p_l} \geq 0,\\
&\frac{1}{3}(3\lambda+2\mu)|\mathrm{div}\mathbf{u}|^2 + 2b_l\mathrm{Re}(\varphi_l\mathrm{div}\overline{\mathbf{u}}) + c_{lj}\varphi_j\overline{\varphi_l} \geq 0.
\end{aligned} \tag{24}$$

It is clear that from inequalities (24), we can deduce that

$$\mathrm{Re}W^{(1)}(\mathbf{U},\mathbf{u})+\mathrm{Re}W^{(2)}(\mathbf{U},\boldsymbol{\varphi})-\frac{1}{\omega}\mathrm{Im}W^{(3)}(\mathbf{U},\mathbf{p})\geq 0. \tag{25}$$

At the same time, based on inequality (25), it follows from (22) that

$$W_0(\mathbf{u})=0,\qquad \frac{1}{3}(3\lambda+2\mu)|\mathrm{div}\mathbf{u}|^2+2b_l\mathrm{Re}(\varphi_l\mathrm{div}\overline{\mathbf{u}})+c_{lj}\varphi_j\overline{\varphi_l}=0. \tag{26}$$

By further applying the conditions of this Theorem, we obtain from (26) that

$$\begin{aligned}&\mathrm{div}\mathbf{u}(\mathbf{x})=0,\qquad \frac{\partial u_j}{\partial x_l}+\frac{\partial u_l}{\partial x_j}=0,\qquad \frac{\partial u_l}{\partial x_l}-\frac{\partial u_j}{\partial x_j}=0,\\ &\varphi_l(\mathbf{x})=0,\qquad l,j=1,2,3\end{aligned} \tag{27}$$

for $\mathbf{x}\in\Omega^+$. Using the first, second, and third equations of (27), we derive that $\mathbf{u}$ is a rigid displacement vector and has the following form

$$\mathbf{u}(\mathbf{x})=\tilde{\mathbf{a}}+\tilde{\mathbf{b}}\times\mathbf{x}\quad\text{for}\quad\mathbf{x}\in\Omega^+, \tag{28}$$

where $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{b}}$ are arbitrary three-component constant vectors, $\tilde{\mathbf{b}}\times\mathbf{x}$ is the vector product of the vectors $\tilde{\mathbf{b}}$ and $\mathbf{x}$. In view of homogeneous boundary condition (21) from (28) we get $\mathbf{u}(\mathbf{x})\equiv\mathbf{0}$ for $\mathbf{x}\in\Omega^+$.

On the other hand, in view of (18) relations we can easily verify that

$$\begin{aligned}&\mathrm{Im}W^{(1)}(\mathbf{U},\mathbf{u})=b_l\mathrm{Im}(\varphi_l\mathrm{div}\overline{\mathbf{u}})-\beta_l\mathrm{Im}(p_l\mathrm{div}\overline{\mathbf{u}}),\\ &\mathrm{Im}W^{(2)}(\mathbf{U},\boldsymbol{\varphi})=b_l\mathrm{Im}(\mathrm{div}\mathbf{u}\overline{\varphi_l})-m_{lj}\mathrm{Im}(p_j\overline{\varphi}_l),\\ &\mathrm{Re}W^{(3)}(\mathbf{U},\mathbf{p})=k_{lj}\nabla p_j\cdot\nabla p_l+\omega\beta_l\mathrm{Im}(\mathrm{div}\mathbf{u}\overline{p}_l)+\omega m_{lj}\mathrm{Im}(\overline{p}_l\varphi_j)+d_{lj}p_j\overline{p}_l\end{aligned}$$

and consequently, we can write

$$\begin{aligned}&-\omega[\mathrm{Im}W^{(1)}(\mathbf{U},\mathbf{u})+\mathrm{Im}W^{(2)}(\mathbf{U},\boldsymbol{\varphi})]+\mathrm{Re}W^{(3)}(\mathbf{U},\mathbf{p})\\ &\qquad=k_{lj}\nabla p_j\cdot\nabla p_l+\gamma_1|p_1-p_2|^2+\gamma_2|p_1-p_3|^2+\gamma_3|p_2-p_3|^2.\end{aligned} \tag{29}$$

Based on conditions of this Theorem and the relation (29), from (22) we have

$$k_{lj}\nabla p_j\cdot\nabla p_l=0. \tag{30}$$

Since, $(k_{lj})_{3\times3}$ is positive definite matrix, from (30) we obtain

$$p_l(\mathbf{x})=c_l=\text{constant}\qquad\text{for}\qquad\mathbf{x}\in\Omega^+,\qquad l=1,2,3.$$

By virtue of (21), we have

$$p_l(\mathbf{x})=0\qquad\text{for}\qquad\mathbf{x}\in\Omega^+,\qquad l=1,2,3.$$

Therefore, $\mathbf{U}(\mathbf{x})=\mathbf{0}$ for $\mathbf{x}\in\Omega^+$. □

Theorem 4. *If $\mu > 0$, $(\zeta_{lj})_{4\times4}$ and $(\alpha_{lj})_{3\times3}$ are the positive definite matrices, $(a_{lj})_{3\times3}$ is a non-negative definite matrix, then two regular solutions of the internal BVP $(II)^{+}_{\mathbf{F},\mathbf{f}}$ may differ only for an additive vector $\mathbf{U} = (\mathbf{u}, \boldsymbol{\varphi}, \mathbf{p})$, where $\boldsymbol{\varphi}$ and $\mathbf{p}$ satisfy the condition*

$$\boldsymbol{\varphi}(\mathbf{x}) = \mathbf{p}(\mathbf{x}) = \mathbf{0}, \qquad \mathbf{x} \in \Omega^{+},$$

the vector $\mathbf{u}$ is a rigid displacement vector and has the form (28).

Proof. Suppose that there are two regular solutions of problem $(II)^{+}_{\mathbf{F},\mathbf{f}}$. Then their difference $\mathbf{U}$ is a regular solution of the internal homogeneous BVP $(II)^{+}_{\mathbf{0},\mathbf{0}}$. Hence, $\mathbf{U}$ is a regular solution of the homogeneous equation (21) for $\mathbf{x} \in \Omega^{+}$, satisfying the homogeneous boundary condition

$$\{\mathbf{R}(\mathbf{D}_{\mathbf{z}}, \mathbf{n})\mathbf{U}(\mathbf{z})\}^{+} = \mathbf{0} \qquad \text{for} \quad \mathbf{z} \in S.$$

Following the same approach as in Theorem 3, we derived (23) relation, from which, based on the conditions of Theorem 4 and relation (22), we have

$$\begin{aligned} &W_0(\mathbf{u}) = 0, \\ &\frac{1}{3}(3\lambda + 2\mu)|\mathrm{div}\mathbf{u}|^2 + 2b_l \mathrm{Re}(\varphi_l \mathrm{div}\overline{\mathbf{u}}) + c_{lj}\varphi_j\overline{\varphi_l} = 0, \\ &\alpha_{lj}p_j\overline{p_l} = 0. \end{aligned} \tag{31}$$

Finally, based on the first and second equations of (31), we derive (27). The first three relations of (27) imply that $\mathbf{u}$ is a rigid displacement vector and has the form (28). From the fourth equation of (27) we have $\boldsymbol{\varphi}(\mathbf{x}) = \mathbf{0}$. In addition the third equation of (31) gives that $\mathbf{p}(\mathbf{x}) = \mathbf{0}$, for $\mathbf{x} \in \Omega^{+}$. Therefore, $\mathbf{U}(\mathbf{x}) = \mathbf{0}$ for $\mathbf{x} \in \Omega^{+}$. □

Theorem 5. *If the requirements of Theorem 3 are met, then the external BVP $(K)^{-}_{\mathbf{F},\mathbf{f}}$ has one regular solution, where $K = I, II$.*

Proof. Suppose that there are two regular solutions of problem $(K)^{-}_{\mathbf{F},\mathbf{f}}$, $K = I, II$. Then their difference $\mathbf{U}$ is a regular solution of the external homogeneous BVP $(K)^{+}_{\mathbf{0},\mathbf{0}}$. Hence, $\mathbf{U}$ is a regular solution of the homogeneous equation (20) for $\mathbf{x} \in \Omega^{-}$, satisfying the homogeneous boundary condition

$$\{\mathbf{U}(\mathbf{z})\}^{-} = \mathbf{0} \tag{32}$$

for $K = I$ and

$$\{\mathbf{R}(\mathbf{D}_{\mathbf{z}},\mathbf{n})\mathbf{U}(\mathbf{z})\}^{-}=\mathbf{0} \tag{33}$$

for $K=II$.

Clearly, by virtue of (18), (32), (33) from (15) we obtain

$$\int_{\Omega^-}W^{(1)}(\mathbf{U},\mathbf{u})d\mathbf{x}=0,\quad \int_{\Omega^-}W^{(2)}(\mathbf{U},\boldsymbol{\varphi})d\mathbf{x}=0,\quad \int_{\Omega^-}W^{(3)}(\mathbf{U},\mathbf{p})d\mathbf{x}=0. \tag{34}$$

Similarly to Theorem 3, from (34) we obtain the following relations:

$$\mathbf{u}(\mathbf{x})=\tilde{\mathbf{a}}+\tilde{\mathbf{b}}\times\mathbf{x},\qquad \boldsymbol{\varphi}(\mathbf{x})=\mathbf{0},\qquad \mathbf{p}(\mathbf{x})=c_l=\text{const.}\quad \text{for}\quad \mathbf{x}\in\Omega^-, \tag{35}$$

where $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{b}}$ are arbitrary three-component constant vectors. In view of the condition at infinity (11) from (35) we get $\mathbf{u}(\mathbf{x})=\mathbf{p}(\mathbf{x})=\mathbf{0}$ for $\mathbf{x}\in\Omega^-$. Thus, we have desired result. □

6 Concluding Remarks

1. In this paper the basic internal and external BVPs of steady vibrations in the coupled linear quasi-static theory of elasticity for nanomaterials with triple porosity are investigated and the following results are obtained:

i) Green's identities are established in the considered theory;

ii) The uniqueness theorems for classical solutions of the aforementioned BVPs are proved.

2. On the basis of results of this paper are possible to investigate:

i) The existence of classical solutions of the BVPs of steady vibrations in the considered theory using the potential method (boundary integral equation method) and the theory of singular integral equations;

ii) The BVPs in the coupled linear quasi-static theory of thermoelasticity for nanomaterials with triple porosity by using the potential method;

iii) The BVPs in the coupled linear theories of elasticity and thermoelasticity for nanomaterials with quadruple porosity.

Compliance with ethical standards

Conflict of Interest: The author declares and assures that no conflict exists.

Acknowledgements

This work is supported by Shota Rustaveli National Science Foundation of Georgia (SRNSFG)[YS-24-148].

REFERENCES

[1] M.A. BIOT (1941) General theory of three-dimensional consolidation. *Journal of Applied Physics* **12** 155-164.

[2] R.K. WILSON, E.C. AIFANTIS (1982) On the theory of consolidation with double porosity. *International Journal of Engineering Science* **20** 1009-1035.

[3] A.H.D. CHENG (2016) Poroelasticity. In: "Theory and Applications of Transport in Porous Media", vol. 27, Switzerland: Springer Int Publ.

[4] H.F. WANG (2000) "Theory of Linear Poro-Elasticity with Applications to Geomechanics and Hydrogeology". Princeton: Princeton Univ. Press.

[5] A.P.S. SELVADURAI, A. SUVOROV (2017) "Thermo-Poroelasticity and Geomechanics". Cambridge: Cambridge University Press.

[6] O. STEPHANSON, L. JING, C.F. TSANG (1996) "Coupled Thermo-Hydro-Mechanical Processes of Fractured Media: Mathematical and Experimental Studies". Elsevier, Amsterdam, Lausanne, New York.

[7] J.R. BOOKER, C. SAVVIDOU (1984) Consolidation around a spherical heat source. *International Journal of Solids and Structures* **20** 1079-1090.

[8] W. DERSKI, S. KOWALSKI (1979) Equations of linear thermoconsolidation. *Archives of Mechanics* **31** 303-316.

[9] M. KURASHIGE (1989) A thermoelastic theory of fluid-filled porous materials. *International Journal of Solids and Structures* **25** 1039-1052.

[10] D.F. MCTIGUE (1986) Thermoelastic response of fluid-saturated porous rock. *Journal of Geophysical Research: Solid Earth* **91**(B9) 9533-9542.

[11] T.S. NGUYEN, A.P.S. SELVADURAI (1995) Coupled thermal-mechanical-hydrological behaviour of sparsely fractured rock: Implications for nuclear fuel waste disposal. *International Journal of Rock Mechanics and Mining Sciences* **32** 465-479.

[12] S. DIEBELS (1999) A micropolar theory of porous media: constitutive modelling. *Transport in Porous Media* **34** 193-208.

[13] B. STRAUGHAN (2008) "Stability and Wave Motion in Porous Media". Springer, New York.

[14] M. SVANADZE (2019) "Potential Method in Mathematical Theories of Multi-Porosity Media". Interdisciplinary Applied Mathematics, vol. 51. Springer Switzerland.

[15] J.W. NUNZIATO, S.C. COWIN (1979) A nonlinear theory of elastic materials with voids. *Archive for Rational Mechanics and Analysis* **72** 175-201.

[16] M. CIARLETTA, D. IEŞAN (1993) "Non-Classical Elastic Solids". Longman Scientific and Technical, John Wiley & Sons, Inc. New York, NY, Harlow, Essex, UK.

[17] M. CIARLETTA, F. PASSARELLA, M. SVANADZE (2014) Plane waves and uniqueness theorems in the coupled linear theory of elasticity for solids with double porosity. *Journal of Elasticity* **114** 55-68.

[18] B. STRAUGHAN (2013) Stability and uniqueness in double porosity elasticity. *International Journal of Engineering Science* **65** 1-8.

[19] B. STRAUGHAN (2017) "Mathematical Aspects of Multi-Porosity Continua". Advances in Mechanics and Mathematics, vol. 38. Springer Inter. Publ. AG, Switzerland.

[20] D. IEŞAN, R. QUINTANILLA (2014) On a theory of thermoelastic materials with a double porosity structure *Journal of Thermal Stresses* **37** 1017-1036.

[21] Z. LIU (2018) "Multiphysics in Porous Materials". Springer Int. Publ. AG, Switzerland.

[22] O. STEPHANSON, J.A. HUDSON, L. JING (2004) "Coupled Thermo-Hydro-Mechanical-Chemical Processes in Geo-Systems: Fundamentals, Modelling, Experiments and Applications". Elsevier, Amsterdam, Boston, Heidelberg, London.

[23] M. SVANADZE (2020) Potential Method in the coupled linear theory of porous elastic solids. *Mathematics and Mechanics of Solids* **25**(3) 768-790.

[24] M. SVANADZE (2019) Boundary integral equations method in the coupled theory of thermoelasticity for porous materials. In: *Proceedings of ASME. IMECE2019*. Volume 9: Mechanics of Solids, Structures, and Fluids, V009T11A033.

[25] L. BITSADZE (2020) Explicit solution of the Dirichlet boundary value problem of elasticity for porous infinite strip. *Zeitschrift fur Angewandte Mathematik und Physik* **71** 145.

[26] L. BITSADZE (2021) Explicit solutions of quasi-static problems in the coupled theory of poroelasticity. *Continuum Mechanics and Thermodynamics* **33** 2481–2492.

[27] M. MIKELASHVILI (2020) Quasi-static problems in the coupled linear theory of elasticity for porous materials. *Acta Mechanica* **231** 877-897.

[28] M. MIKELASHVILI (2021) Quasi-static problems in the coupled linear theory of thermoporoelasticity. *Journal of Thermal Stresses* **44** 236-259.

[29] M.M. SVANADZE (2021) Potential method in the coupled theory of viscoelasticity of porous materials. *Journal of Elasticity* **144** 119-140.

[30] M. SVANADZE (2021) Potential Method in the coupled theory of elastic double-porosity materials. *Acta Mechanica* **232** 2307-2329.

[31] M.M. SVANADZE (2021) Problems of steady vibrations in the coupled linear theory of double-porosity viscoelastic materials. *Archives of Mechanics* **73** 365-390.

[32] M. MIKELASHVILI (2023) Potential method in the coupled linear quasi-static theory of elasticity for double porosity materials. *Transactions of A. Razmadze Mathematical Institute* **177**(3) 431-450.

[33] M. MIKELASHVILI (2023) Potential method in the coupled linear quasi-static theory of thermoelasticity for double porosity materials. *Archives of Mechanics* **75** 1-32.

[34] V.D. KUPRADZE (1965) "Potential Methods in the Theory of Elasticity". Israel Program Sci. Transl., Jerusalem.

[35] M. SVANADZE (2022) Potential method in the coupled linear theory of elasticity for materials with triple porosity. *Transactions of A. Razmadze Mathematical Institute* **176**(1) 83-98.

[36] V.D. KUPRADZE, T.G. GEGELIA, M.O. BASHELEISHVILI, T.V. BURCHULADZE (1979) "Three-Dimensional Problems of the Mathematical Theory of Elasticity and Thermoelasticity". North-Holland, Amsterdam, New York, Oxford.