Journal of Theoretical and Applied Mechanics, Sofia, Vol.55 (2025) pp. 371-384
GENERAL MECHANICS

A GENERAL ANALYTICAL STABILITY ANALYSIS OF THE MATHIEU-HILL EQUATION

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[Received: 17 January 2025. Accepted: 30 September 2025]

doi: https://doi.org/10.55787/jtams.25.55.3.371

ABSTRACT: A general Mathieu-Hill equation is considered. The periodic function of the equation consists of series of cosine functions with arbitrary number of elements. Stability analysis is performed via the strained parameters method, a perturbation technique. Stability branches are calculated approximately up to second order for the first, second and general n'th regions for an arbitrary number of elements m. Hence, the analysis presents a general solution for any n'th stability region with arbitrarily large m number of elements which cannot be traced in the past literature. The first three stability regions are plotted for a special problem having three parametric excitation terms. Numerical simulations of sample stable, and unstable solutions are given. As with all classical perturbation methods, the analytical solutions for stability branches are compatible with the real stability branches if the perturbation parameter is not large. For large perturbation parameters, the real stability branches and the approximate analytical ones diverge from each other making the approximate stability branches not reliable. This general approach in which there are infinite number of parametric excitations and general solutions for any stability branches is the main contribution of this work to the literature. The work has the potential of application to a wide range of problems modeled by the Mathieu-Hill equations. The general formulas derived are applied to a specific rotating blade vibration problem arising in rotor dynamics and turbomachinery. A study that results in a stability map for the shaft/blade design and operational parameters is extremely valuable for designers, diagnostics, and maintenance engineers. The system contains a number of even frequency parametric excitations. Special solutions of the previous results and the reduction of the general solutions presented here to this specific problem are compatible within the range of order of magnitudes considered.

KEY WORDS: Mathieu Equation; Mathieu-Hill Equation; Stability Analysis; Strutt Diagrams; Strained Parameters Method.

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1 Introduction

Mathieu-Hill equation and its special form, Mathieu equation, find applications in vast areas of physical problems. Mathieu first derived the equation when studying the vibration of elliptical membranes [1]. The inverted pendulum problem, radio frequency quadrupole, frequency modulation in microphones, stability of a floating body, the Paul trap for charged particles, the mirror trap for neutral particles are some examples of problems reducing to Mathieu type equations [2]. Stability of ships and railroad trains, resonant internal sensors [3], propagation of electromagnetic waves, distributed feedback lasers, fiber-optic wave guides [4] are application areas of the equations. The vibrations of Euler beams subjected to unsteady wind force reduces to a Mathieu-Hill equation [5]. The equation appears in the parametrically excited composite plates [6]. Harmonically excited nanobeam problem [7], vibrations of carbon nanotubes [8] reduces to Mathieu-Hill equations. A double frequency Mathieu equation with the frequency ratio not an integer was considered in determining the effects of time dependent gravity on the thermal convective stability of a fluid layer [9]. In the discretization process of the vibrations of an axially accelerating string problem, the single mode discretization reduced to the Mathieu equation [10]. The rotating blade vibration due to torsional excitation yielded a special Mathieu-Hill equation [11]. For torsional vibrations and stability analysis, see also [12, 13].

Mathieu-Hill equations and their special cases of Mathieu equation are well known examples of parametrically excited systems. Depending on the physical parameters, the solutions may be decaying (stable), bounded or unbounded (unstable) within time. Usually stability charts (Strutt diagrams) are formed for the specific problem of technological interest to show the stable and unstable regions. The lines separating the stable and unstable regions are called as the stability branches. The well-known analytical technique is the perturbation method which enables the calculation of stability branches analytically. For calculation of the stability branches by various perturbation techniques such as the Strained Parameters Method, the Whittaker's Method, the Method of Multiple Scales and the Method of Averaging, see [14–17]. The well-known numerical technique for calculating the stability of Mathieu type equations is the Floquet theory used in [8, 10, 11, 14] which is based on one period integration of the equation, calculating the so-called Monodromy matrix and determining the stability through the eigenvalues of the matrix. Forming a stability chart via numerical methods however requires extensive computational time [11] because for each point in the parametric space, the equation has to be integrated over one period and if one needs a high resolution of such charts, the computational effort tremendously increases. The approximate analytical techniques have the advantage of directly determining the stability branches with the deficiency of being confined to small perturbation parameters. Recently, there are some attempts to validate the analytical results for large perturbation parameters also [18–20] in which the Homotopy Perturbation Method [18, 19] and the Algebraic Harmonic Balance Method [20] were employed in the calculations. For a theoretical analysis of the non-oscillatory solutions of the Mathieu equation with two frequencies, see [21].

In this work, a Mathieu-Hill equation with arbitrarily large numbers of harmonic functions is considered. Stability branches are analytically determined in their most general form by the strained parameters method, a perturbation technique. The first and second stable regions are calculated first. Then the n'th stable region with m number of harmonic functions, n and m being arbitrarily large, is calculated. Previous studies investigated a restricted form of the Mathieu-Hill equation, considering a few harmonic excitation terms. In contrast, in this study, a generalized form with infinitely many harmonics case is treated in its most general form. Specifically, formulas for the boundaries of the n'th stability region for arbitrary m term harmonic excitations are reported in this study for the first time. The algorithm developed may be applicable to a wide range of physical problems as special cases. One such application, namely the rotating blade vibration problem is treated as an example of the implementation of the general solution procedure.

2 The Equation of Motion and the Strained Parameters Method

Consider the Mathieu-Hill equation with arbitrary number of harmonic parametric excitation terms

$$\ddot{u} + \left(\delta + \varepsilon \sum_{k=1}^{m} \alpha_k \cos 2kt\right) u = 0\,. \tag{1}$$

The equation is a generalization of many Mathieu-Hill type equations. For example, the model given in [11] is a special case of (1) with $m = 4$ and appropriate definitions of α_{1-4}. $u = u(t)$ is the time dependent response function to the multi-frequency parametric excitation. The equation is assumed to be cast into a non-dimensional form.

The Strained Parameters Method is one of the most effective perturbation methods when handling such problems [14–17]. Depending on the locations in the $\delta - \varepsilon$ plane, the solutions may be stable or unstable. The aim in the perturbation approach is to determine the stability branches separating the stable and unstable regions. Assuming an expansion up to second order

$$u\,(t) = u_0\,(t) + \varepsilon u_1\,(t) + \varepsilon^2 u_2\,(t) + \cdots \tag{2}$$

and since the straightforward expansion breaks down for $\delta \approx n^2$, $n = 0, 1, 2, 3, \ldots$, an expansion of the parameter δ is necessary in the vicinity of such values to eliminate the so-called secular terms responsible for the non-physical terms growing in time

$$\delta = n^2 + \varepsilon\delta_1 + \varepsilon^2\delta_2 + \cdots\,, \qquad n = 0, 1, 2, \ldots \tag{3}$$

Substituting the expansions into the original equation and separating terms with respect to orders yields

$$O(1) : \ddot{u}_0 + n^2 u_0 = 0\,, \tag{4}$$

$$O(\varepsilon) : \ddot{u}_1 + n^2 u_1 = -\delta_1 u_0 - u_0 \sum_{k=1}^{m} \alpha_k \cos 2kt\,, \tag{5}$$

$$O(\varepsilon^2) : \ddot{u}_2 + n^2 u_2 = -\delta_2 u_0 - \delta_1 u_1 - u_1 \sum_{k=1}^{m} \alpha_k \cos 2kt\,. \tag{6}$$

The first order solution is

$$u_0 = a\cos(nt) + b\sin(nt)\,, \qquad n = 0, 1, 2, \ldots \tag{7}$$

Solutions vary depending on the number of n. $n = 1$, $n = 2$ and $n = n$ cases will be treated separately.

3 The First Unstable Region

Due to its importance, the first instability region is treated first. For $n = 1$, the first order solution is

$$u_0 = a\cos t + b\sin t\,. \tag{8}$$

Substituting this solution to the next order of approximation and arranging

$$\begin{aligned}\ddot{u}_1 + u_1 =& \cos t\left(-\delta_1 - \frac{\alpha_1}{2}\right)a + \sin t\left(-\delta_1 + \frac{\alpha_1}{2}\right)b \\ &- \sum_{k=1}^{m} \alpha_k \left(\frac{a}{2}\cos(2k+1)\,t + \frac{b}{2}\sin(2k+1)\,t\right) \\ &- \sum_{k=2}^{m} \alpha_k \left(\frac{a}{2}\cos(2k-1)\,t - \frac{b}{2}\sin(2k-1)t\right).\end{aligned} \tag{9}$$

Two conditions arise from the requirement of elimination of secular terms

$$\left(\delta_1 + \frac{\alpha_1}{2}\right)a = 0 \qquad \text{and} \qquad \left(\delta_1 - \frac{\alpha_1}{2}\right)b = 0\,. \tag{10}$$

Satisfaction of both equations require then either $\delta_1 = -\alpha_1/2$, $b = 0$ or $\delta_1 = \alpha_1/2$, $a = 0$. The two cases lead to two different stability branches and should be treated separately

(I) $\boldsymbol{\delta_1 = -\alpha_1/2, \ \ b = 0}$

The equation to be solved is from (9)

$$\ddot{u}_1 + u_1 = -\sum_{k=1}^{m} \alpha_k \frac{a}{2} \cos(2k+1)\,t - \sum_{k=2}^{m} \alpha_k \frac{a}{2} \cos(2k-1)\,t\,, \tag{11}$$

the solution of which is

$$u_1 = \frac{a}{2}\Big(\sum_{k=1}^{m} \frac{\alpha_k}{(2k+1)^2-1}\cos(2k+1)t + \sum_{k=2}^{m} \frac{\alpha_k}{(2k-1)^2-1}\cos(2k-1)t\Big). \tag{12}$$

Substituting this solution to the next level of approximation yields

$$\begin{aligned} \ddot{u}_2 + u_2 = a\cos t\Bigg(-\delta_2 - \frac{1}{4}\Big\{\sum_{k=1}^{m} \frac{\alpha_k(\alpha_k+\alpha_{k+1})}{(2k+1)^2-1} \\ + \sum_{k=2}^{m} \frac{\alpha_k(\alpha_k+\alpha_{k-1})}{(2k-1)^2-1}\Big\}\Bigg) + NST\,, \end{aligned} \tag{13}$$

where *NST* stands for non-secular terms. Elimination of the secular term then requires

$$\delta_2 = -\frac{1}{4}\Big\{\sum_{k=1}^{m} \frac{\alpha_k(\alpha_k+\alpha_{k+1})}{(2k+1)^2-1} + \sum_{k=2}^{m} \frac{\alpha_k(\alpha_k+\alpha_{k-1})}{(2k-1)^2-1}\Big\}. \tag{14}$$

One should take $\alpha_{m+1} = 0$ when summing up the terms since this term is not a system parameter.

(II) $\boldsymbol{\delta_1 = \alpha_1/2, \ \ a = 0}$

Proceeding in a similar way,

$$u_1 = \frac{b}{2}\Big(\sum_{k=1}^{m} \frac{\alpha_k}{(2k+1)^2-1}\sin(2k+1)t - \sum_{k=2}^{m} \frac{\alpha_k}{(2k-1)^2-1}\sin(2k-1)t\Big) \tag{15}$$

and upon eliminating the secular term at the last level requires

$$\delta_2 = -\frac{1}{4}\Big\{\sum_{k=1}^{m} \frac{\alpha_k(\alpha_k-\alpha_{k+1})}{(2k+1)^2-1} + \sum_{k=2}^{m} \frac{\alpha_k(\alpha_k-\alpha_{k-1})}{(2k-1)^2-1}\Big\}. \tag{16}$$

In summary, combining the results, the two stability branches emanating from $n = 1$ are

$$\delta = 1 - \varepsilon\frac{\alpha_1}{2} - \frac{\varepsilon^2}{4}\left\{\sum_{k=1}^{m}\frac{\alpha_k(\alpha_k + \alpha_{k+1})}{(2k+1)^2 - 1} + \sum_{k=2}^{m}\frac{\alpha_k(\alpha_k + \alpha_{k-1})}{(2k-1)^2 - 1}\right\}, \tag{17}$$

$$\delta = 1 + \varepsilon\frac{\alpha_1}{2} - \frac{\varepsilon^2}{4}\left\{\sum_{k=1}^{m}\frac{\alpha_k(\alpha_k - \alpha_{k+1})}{(2k+1)^2 - 1} + \sum_{k=2}^{m}\frac{\alpha_k(\alpha_k - \alpha_{k-1})}{(2k-1)^2 - 1}\right\}. \tag{18}$$

For $m = 1$, $\alpha_1 = 2$, the equation is the Mathieu equation treated in [14] and the stability branches reduce to

$$\delta = 1 - \varepsilon - \frac{1}{8}\varepsilon^2; \qquad \delta = 1 + \varepsilon - \frac{1}{8}\varepsilon^2, \tag{19}$$

which is the same result reported there. For the special case of $m = 3$, the stability branches are

$$\delta = 1 - \varepsilon\frac{\alpha_1}{2} - \frac{\varepsilon^2}{192}\left(6\alpha_1^2 + 8\alpha_2^2 + 3\alpha_3^2 + 12\alpha_1\alpha_2 + 4\alpha_2\alpha_3\right), \tag{20}$$

$$\delta = 1 + \varepsilon\frac{\alpha_1}{2} - \frac{\varepsilon^2}{192}\left(6\alpha_1^2 + 8\alpha_2^2 + 3\alpha_3^2 - 12\alpha_1\alpha_2 - 4\alpha_2\alpha_3\right). \tag{21}$$

For $\alpha_1 > 0$, equation (20) represents the left branch and (21) represents the right branch. For $\alpha_1 < 0$, the reverse is true and (20) represents the right and (21) the left branch.

A sample plot of stability branches is given in Figure 1. By using the functional relationships given in equations (20) and (21), the $\delta - \varepsilon$ curves can easily be plotted. In this study, all figures are obtained by using the MATLAB program. The region between the branches is unstable and the regions on the left and right of the branches are stable. The solutions on the branches are harmonic bounded solutions as determined by the perturbation analysis. Perturbation solutions must be uniform for validity. Uniformity requires that the correction terms be much smaller than the leading terms. For the specific stability branch solution of (20) for example, the uniformity conditions are

$$\left|\varepsilon\frac{\alpha_1}{2}\right| \ll 1, \qquad \left|\frac{\varepsilon^2}{192}(6\alpha_1^2 + 8\alpha_2^2 + 3\alpha_3^2 - 12\alpha_1\alpha_2 - 4\alpha_2\alpha_3)\right| \ll \left|\varepsilon\frac{\alpha_1}{2}\right|, \tag{22}$$

which depends on the perturbation parameters as well as the other coefficients.

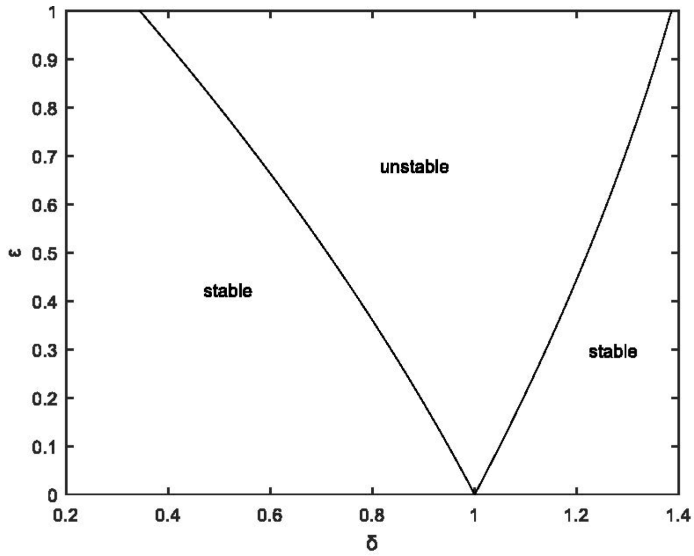

Fig. 1. Stability branches for the first unstable region ($n = 1$, $\alpha_1 = 1$, $\alpha_2 = 1$, $\alpha_3 = -2$).

4 The Second Unstable Region

For $n = 2$, the first order solution is

$$u_0 = a\cos 2t + b\sin 2t \tag{23}$$

Substituting this solution to the next order of approximation and arranging yields

$$\begin{aligned}\ddot{u}_1 + 4u_1 = {} & \cos 2t\left(-\delta_1 - \frac{\alpha_2}{2}\right)a + \sin 2t\left(-\delta_1 + \frac{\alpha_2}{2}\right)b - \frac{\alpha_1}{2}a \\ & - \sum_{k=1}^{m}\alpha_k\left(\frac{a}{2}\cos\left(2k+2\right)t + \frac{b}{2}\sin\left(2k+2\right)t\right) \\ & - \sum_{k=3}^{m}\alpha_k\left(\frac{a}{2}\cos\left(2k-2\right)t - \frac{b}{2}\sin(2k-2)t\right).\end{aligned} \tag{24}$$

Elimination of secular terms requires

$$\left(\delta_1 + \frac{\alpha_2}{2}\right)a = 0 \qquad \text{and} \qquad \left(\delta_1 - \frac{\alpha_2}{2}\right)b = 0\,. \tag{25}$$

The two cases $\delta_1 = -\alpha_2/2$, $b = 0$ or $\delta_1 = \alpha_2/2$, $a = 0$ should be treated separately

(I) $\boldsymbol{\delta_1 = -\alpha_2/2, \;\; b = 0}$

The equation to be solved is from (24)

$$\ddot{u}_1 + 4u_1 = -\frac{\alpha_1}{2}a - \frac{a}{2}\left(\sum_{k=1}^{m} \alpha_k \cos(2k+2)t\right) + \sum_{k=3}^{m} \alpha_k \cos(2k-2)t\,, \tag{26}$$

the solution of which is

$$u_1 = -\alpha_1 \frac{a}{8} + \frac{a}{2}\left(\sum_{k=1}^{m} \frac{\alpha_k \cos(2k+2)t}{(2k+2)^2 - 4} + \sum_{k=3}^{m} \frac{\alpha_k \cos(2k-2)t}{(2k-2)^2 - 4}\right). \tag{27}$$

Substituting this solution to the next level of approximation and elimination of secular terms yield

$$\delta_2 = \frac{1}{8}\alpha_1^2 - \frac{1}{4}\left\{\sum_{k=1}^{m} \frac{\alpha_k(\alpha_k + \alpha_{k+2})}{(2k+2)^2 - 4} + \sum_{k=3}^{m} \frac{\alpha_k(\alpha_k + \alpha_{k-2})}{(2k-2)^2 - 4}\right\}. \tag{28}$$

One should take $\alpha_{m+1} = \alpha_{m+2} = 0$ when summing up the terms since they are not system parameters. Also if $m < 3$ ignore the second summation term.

(II) $\boldsymbol{\delta_1 = \alpha_2/2, \;\; a = 0}$

Proceeding in a similar way,

$$u_1 = \frac{b}{2}\left(\sum_{k=1}^{m} \frac{\alpha_k \sin(2k+2)t}{(2k+2)^2 - 4} - \sum_{k=3}^{m} \frac{\alpha_k \sin(2k-2)t}{(2k-2)^2 - 4}\right) \tag{29}$$

and upon eliminating the secular term at the last level requires

$$\delta_2 = -\frac{1}{4}\left\{\sum_{k=1}^{m} \frac{\alpha_k(\alpha_k - \alpha_{k+2})}{(2k+2)^2 - 4} + \sum_{k=3}^{m} \frac{\alpha_k(\alpha_k - \alpha_{k-2})}{(2k-2)^2 - 4}\right\}. \tag{30}$$

Therefore, the two stability branches emanating from $n = 2$ are

$$\delta = 4 - \varepsilon\frac{\alpha_2}{2} + \frac{\varepsilon^2}{4}\left\{\frac{1}{2}\alpha_1^2 - \left(\sum_{k=1}^{m} \frac{\alpha_k(\alpha_k + \alpha_{k+2})}{(2k+2)^2 - 4} + \sum_{k=3}^{m} \frac{\alpha_k(\alpha_k + \alpha_{k-2})}{(2k-2)^2 - 4}\right)\right\}, \tag{31}$$

$$\delta = 4 + \varepsilon\frac{\alpha_2}{2} - \frac{\varepsilon^2}{4}\left\{\sum_{k=1}^{m} \frac{\alpha_k(\alpha_k - \alpha_{k+2})}{(2k+2)^2 - 4} + \sum_{k=3}^{m} \frac{\alpha_k(\alpha_k - \alpha_{k-2})}{(2k-2)^2 - 4}\right\}. \tag{32}$$

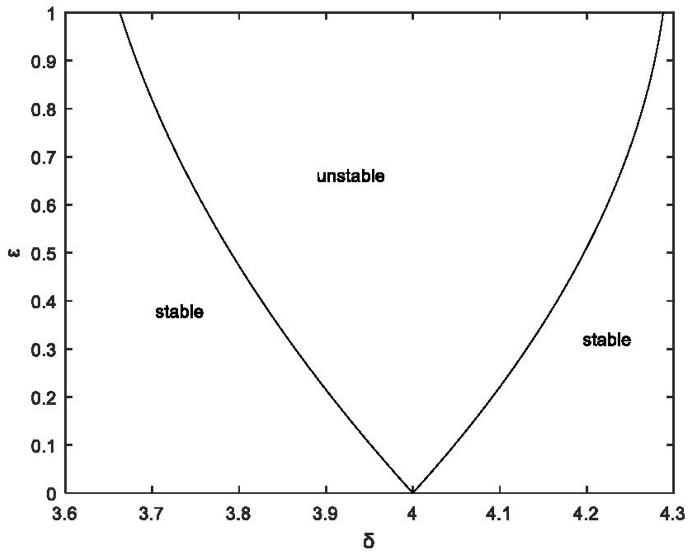

Fig. 2. Stability branches for the second unstable region ($n = 2$, $\alpha_1 = 1$, $\alpha_2 = 1$, $\alpha_3 = -2$).

For $m = 1$, $\alpha_1 = 2$, the solution reduces to

$$\delta = 4 + \frac{5}{12}\varepsilon^2\,; \qquad \delta = 4 - \frac{1}{12}\varepsilon^2\,, \tag{33}$$

which is the same result reported in [14]. For the special case of $m = 3$, the stability branches are

$$\delta = 4 - \varepsilon\frac{\alpha_2}{2} + \varepsilon^2\left(\frac{5}{48}\alpha_1^2 - \frac{1}{128}\alpha_2^2 - \frac{1}{240}\alpha_3^2 - \frac{1}{24}\alpha_1\alpha_3\right), \tag{34}$$

$$\delta = 4 + \varepsilon\frac{\alpha_2}{2} - \varepsilon^2\left(\frac{1}{48}\alpha_1^2 + \frac{1}{128}\alpha_2^2 + \frac{1}{40}\alpha_3^2 - \frac{1}{24}\alpha_1\alpha_3\right). \tag{35}$$

The unstable region vanishes for $O(\varepsilon)$ when $\alpha_2 = 0$. The $O(\varepsilon^2)$ terms vanishes if $\alpha_3 = 40\alpha_1/31$. The region under these conditions would then be very narrow having $O(\varepsilon^3)$ and smaller terms.

Plot of the stability branches is given in Figure 2 for this case.

5 The General Solution

In fact, any arbitrary instability region $n = n$ can be retrieved from the analysis. The calculations reveal that there are three distinct cases: (i) $n > 2m$, (ii) $m < n \leq 2m$, (iii) $n \leq m$. Skipping the involved algebra and proceeding with a similar way as in the previous sections, the results for each case are directly given in this section.

$$\delta = n^2 - \frac{\varepsilon^2}{4}\sum_{k=1}^{m}\alpha_k^2\left(\frac{1}{(2k+n)^2-n^2}+\frac{1}{(2k-n)^2-n^2}\right) \quad (n>2m) \tag{36}$$

$$\delta = n^2 - \frac{\varepsilon^2}{4}\sum_{k=1}^{m}\frac{\alpha_k^2}{(2k+n)^2-n^2}+\frac{\alpha_k(\alpha_k+\alpha_{n-k})}{(2k-n)^2-n^2} \quad m<n\le 2m \tag{37}$$

$$\delta = n^2 - \frac{\varepsilon^2}{4}\sum_{k=1}^{m}\frac{\alpha_k^2}{(2k+n)^2-n^2}+\frac{\alpha_k(\alpha_k-\alpha_{n-k})}{(2k-n)^2-n^2} \quad m<n\le 2m \tag{38}$$

$$\delta = n^2 - \varepsilon\frac{\alpha_n}{2} - \frac{\varepsilon^2}{4}\Big(\sum_{k=1}^{m}\frac{\alpha_k(\alpha_k+\alpha_{n+k})}{(2k+n)^2-n^2} + \sum_{k=1(k\neq n)}^{m}\frac{\alpha_k(\alpha_k+\alpha_{n-k}+\alpha_{k-n})}{(2k-n)^2-n^2}\Big) \quad n\le m \tag{39}$$

$$\delta = n^2 + \varepsilon\frac{\alpha_n}{2} - \frac{\varepsilon^2}{4}\Big(\sum_{k=1}^{m}\frac{\alpha_k(\alpha_k-\alpha_{n+k})}{(2k+n)^2-n^2} + \sum_{k=1(k\neq n)}^{m}\frac{\alpha_k(\alpha_k-\alpha_{n-k}-\alpha_{k-n})}{(2k-n)^2-n^2}\Big) \quad n\le m \tag{40}$$

When calculating the sums, note that α_i is within the range of $1\le i\le m$. If $i<1$ or $i>m$ take $\alpha_i=0$ in the calculations. For the case of $n>2m$, there is only one stability branch up to $O(\varepsilon^2)$. The left and right stability branches diverge from each other at $O(\varepsilon^3)$. For the case of $m<n\le 2m$, the left and right stability branches diverge from each other at $O(\varepsilon^2)$. Finally, when $n\le m$, the left and right branches diverge at $O(\varepsilon)$. For n =1 and $n=2$, (39) and (40) reduces to the results of the previous sections.

Results reported in [11] can be retrieved from the general stability branches given here by appropriate definitions of the α_k coefficients. The stability branches agree up to $O(\varepsilon^2)$. More details are given in the next section.

For the special case of n =3, m =3, the stability branches (39) and (40) reduces to

$$\delta = 9 - \varepsilon\frac{\alpha_3}{2} + \varepsilon^2\left(\frac{1}{64}\alpha_1^2+\frac{1}{40}\alpha_2^2-\frac{1}{288}\alpha_3^2+\frac{1}{16}\alpha_1\alpha_2\right), \tag{41}$$

$$\delta = 9 + \varepsilon\frac{\alpha_3}{2} + \varepsilon^2\left(\frac{1}{64}\alpha_1^2+\frac{1}{40}\alpha_2^2-\frac{1}{288}\alpha_3^2-\frac{1}{16}\alpha_1\alpha_2\right). \tag{42}$$

The plot of the stability branches is given in Figure 3.

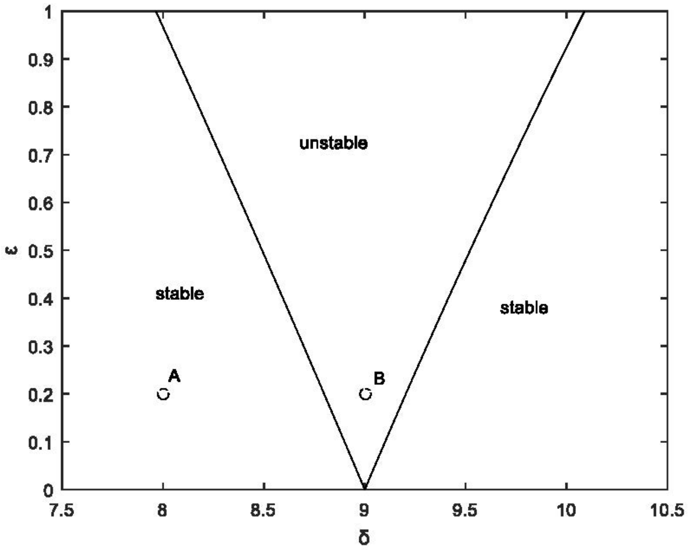

Fig. 3. Stability branches for the third unstable region ($n = 3$, $\alpha_1 = 1$, $\alpha_2 = 1$, $\alpha_3 = -2$).

Sample stable (Point A) and unstable (Point B) points are chosen in the diagram and time histories are given in Figures 4 and 5.

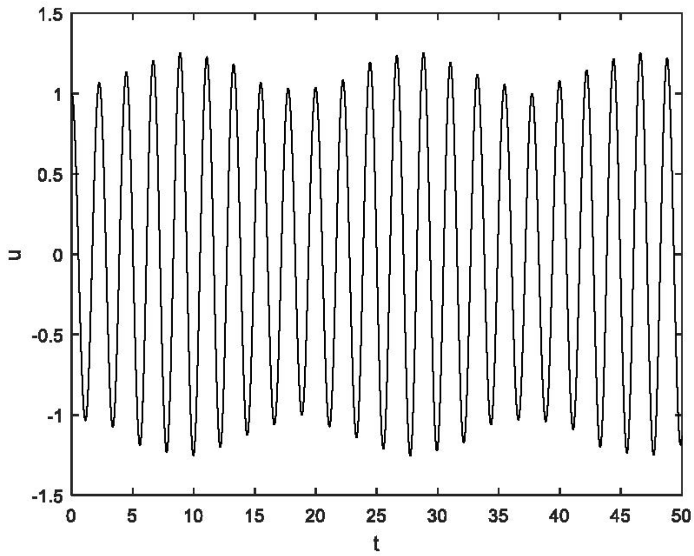

Fig. 4. Time history of the stable point A ($\delta = 8$, $\varepsilon = 0.2$, $\alpha_1 = 1$, $\alpha_2 = 1$, $\alpha_3 = -2$).

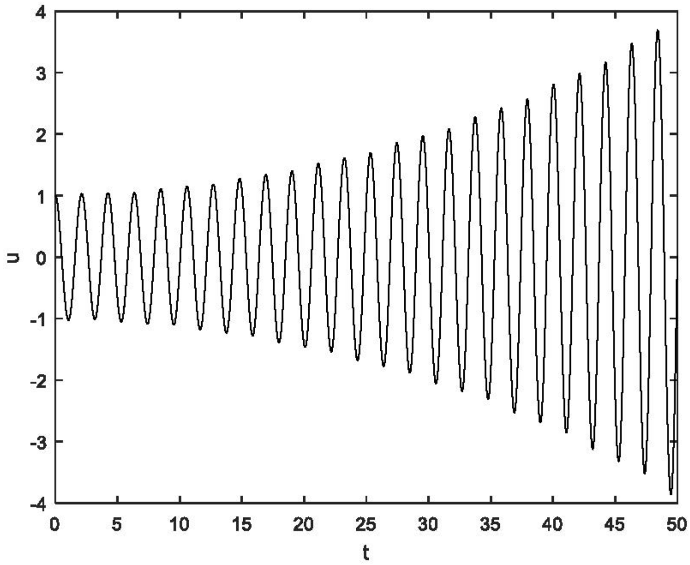

Fig. 5. Time history of the unstable point B ($\delta = 9$, $\varepsilon = 0.2$, $\alpha_1 = 1$, $\alpha_2 = 1$, $\alpha_3 = -2$).

6 An Application to Rotating Blade Vibrations

An application of this study may be the rotating blade vibrations. The mathematical derivation of the equation of motion and reduction to Mathieu-Hill equation was outlined in detail in [11]

$$x'' + \left(\delta + 8\varepsilon r_2 \cos 2\tau + 2\varepsilon^2 \cos 4\tau + \frac{1}{2}\varepsilon^4 \cos 8\tau\right) x = 0 . \tag{43}$$

The parameters and the new time variable are defined as follows:

$$\delta = 4r_1^2 + 4r_2^2 + \frac{3}{2}\varepsilon^2 , \qquad r_1 = \frac{\omega_B}{\omega} , \qquad r_2 = \frac{\Omega}{\omega} , \qquad \tau = \frac{1}{2}\omega t , \tag{44}$$

where r_1 is the ratio of blade natural frequency ω_B to the torsional excitation frequency ω, r_2 is the ratio of the constant rotation speed Ω to the torsional excitation frequency ω and ε is the torsional vibration amplitude which is assumed to be small. δ is defined in terms of the dimensionless parameters indicated.

Comparing (43) with our general equation (1), note that

$$\alpha_1 = 8r_2, \quad \alpha_2 = 2\varepsilon, \quad \alpha_3 = 0, \quad \alpha_4 = \frac{1}{2}\varepsilon^3 \quad . \tag{45}$$

Substituting the coefficients (45) into the stability branches given in (17) and (18), neglecting terms smaller than O(ε^4), the results are

$$\delta = 1 - \varepsilon 4 r_2 - \varepsilon^2 2 r_2^2 - \varepsilon^3 r_2 - \frac{1}{6}\varepsilon^4\,, \tag{46}$$

$$\delta = 1 + \varepsilon 4 r_2 - \varepsilon^2 2 r_2^2 + \varepsilon^3 r_2 - \frac{1}{6}\varepsilon^4\,. \tag{47}$$

Comparing the above results with those given in [11] for the first stability region, the stability boundaries are exactly the same up to O(ε^2). There are some discrepancies between the results for the coefficients of O(ε^3) and O(ε^4) terms. This is because in this analysis, perturbation expansion up to O(ε^2) is taken whereas in that analysis the expansions are carried out up to O(ε^4) terms. Similar conclusions can be drawn for the second, third and fourth stability regions given in that study. Since the torsional vibration amplitudes ε are rather small in practical applications, the discrepancies have negligible effects on the stability boundaries. As noted in [11], the exact location of the stability boundaries requires extensive numerical calculations with high computational costs. The analytical solutions therefore provide a guidance in lowering such costs since one may determine the location of the boundaries approximately by the analytical results.

7 Concluding Remarks

For a general Mathieu-Hill equation having arbitrary numbers of harmonic parametric excitations, a detailed stability analysis is given. By employing the Stained Parameters Method, the stability branches are calculated approximately. Explicit functional forms of the stability branches are given in their most general form for the numbers n and m where n represents the number of the stability region and m represents the number of the parametric excitation terms. The general formalism presented here can be applied to a variety of problems of technological interest in determining their stability. A rotating blade vibration problem has been treated as a direct application of the general formalism developed in this study.

Further studies may be to apply the results to other specific problems. Calculations can be carried to higher order of approximations but this may be a tedious work and the formulas may be much more complicated.

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