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SOLID MECHANICS

EFFECT OF MATERIAL COMPOSITION ON FREE VIBRATION OF BIDIRECTIONAL FUNCTIONALLY GRADED BEAM VIA A QUASI-3D THEORY

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ABSTRACT: In this paper, the influence of material composition on free vibration of bidirectional functionally graded beam using a quasi-3D theory is investigated. The material properties are assumed to be graded in both the thickness and longitudinal directions by power gradation laws and four different distribution patterns are considered. Equations of motion based on a quasi-3D model that contains undetermined integral forms and involves few unknowns to derive, are obtained from Hamilton's principle. The problem is solved using the Navier solution for a simply supported beam. The accuracy of the present solution is demonstrated by comparing it with some results available in the literature and a good agreement is showed. The effects of the type of material distribution, power-law indexes, and the aspect ratio on the fundamental frequencies are examined and highlighted.

KEY WORDS: free vibration; bidirectional functionally graded beam; quasi-3D theory; aspect ratio.

1 INTRODUCTION

Beams are widely used in structural engineering. Homogeneous, composite, functionally graded materials (FGMs) are used in these beams to meet different engineering design requirements including thermal-structural demands. FGMs are a new class of non-homogeneous composites [1,2], and they are commonly used for low thermal conductivity and high fracture toughness. Typically composed of ceramic and metal, with ceramic reducing heat transfer to protect the metal from corrosion and oxidation, while metal provides higher strength and fracture toughness. Recently, functionally graded materials have found various applications in aircrafts and space vehicles, electrical devices, optics, nuclear power plants, engineering, automotive industry and so on [3–5].

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To predict and to understand the behaviors of FGMs structures, a great deal of research has been carried out on the advanced theories and analysis methods. In the literature, many beam theories have been studied, the classical beam theory (CBT), called the Euler–Bernoulli theory, the first order or Timoshenko theory (FSBT), and higher-order beam theories (HSBTs). It should be noted that the Euler beam theory also called classical beam theory (CBT) ignores shear deformation and applies only to slender beams. The Timoshenko beams theory or called the first order beam theory (FSBT) accounts for the shear deformation effect, but requires a shear correction factor. In order to take into account the transverse shear deformation, researchers have developed higher order shear theories for the study of FG beams (HSBT) [6], Bourada et al. [7] studied the bending and vibration behavior of FG beams. Thai and Vo [8] examined the effect of the volume fraction of constituents and shear deformation on the bending and vibration behavior of FG beams employing different higher-order shear-deformation beam theories. Both FSDT and HSDT ignore the effect of the thickness stretching, which is significant in thick beams and plates. A number of Quasi-3D theories have been developed, in which the effects of shear deformation and thickness stretching were included [9, 10].

In order to ensure the requirements of stress and thermal distributions in two or more directions, researchers have studied the bidirectional FG beam models with the volume fraction of constituents varying in both the thickness and longitudinal directions [11–20]. Şimşek [21] studied the free and forced vibration of bi-directional functionally graded (BDFG) beam subjected to a moving load. He used the Timoshenko beam theory as well as Euler beam theory to derive the equations of motion. By utilizing generalized finite element method, Pham [22] studied hygro-thermal vibration of bidirectional functionally graded porous curved beams on variable elastic foundation. The quasi-3D theory and the Symmetric Smoothed Particle Hydrodynamics (SSPH) method have been examined by Karamanli [23] to investigate the static behavior of BDFG sandwich beams subjected to various sets of boundary conditions. Using a quasi-3D theory, Trinh et al. [24], Studied the free vibration behavior of BDFG micro-beams under arbitrary boundary conditions. Size dependent bending analysis of two directional functionally graded microbeams was investigated by Karamanli and Vo [25] employing the same theory and the finite element analysis. A study on the crack presence effect on dynamical behaviour of bi-directional compositionally imperfect material graded micro beams was conducted by Saimi et al. [26].

On the basis of the above summary and to the best authors' experience, no research has been done on the free vibration of BDFG beams with the consideration of the effect of material distribution. In this paper, effect of material composition on free vibration of bidirectional functionally graded beam is studied by using a quasi-3D theory. In this theory, the displacement field defined by introducing the stretching effect without a need for any shear correction factor and contained undetermined integral terms to reduce the number of unknown's functions. The material properties vary along both thickness and axial directions according to a power law by considering four different distribution types. Governing equations of motion are derived from the Hamilton's principle. Navier -type solution for simply-supported beams is developed to solve the problem. Comparative studies are considered to check the accuracy of the present formulation. A detailed parametric study is presented to show the effect of the different parameters on the free vibration of BDFG beam.

2 Mathematical Formulation

Consider a bidirectional functionally graded beam (BDFGB) of thickness h, width b and length L, as shown in the Fig. 1.

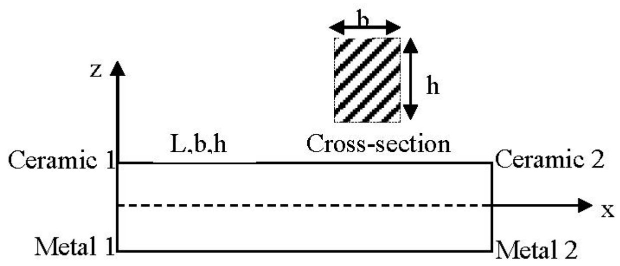

Fig. 1. Geometry and coordinates of BDFG beam.

2.1 Material properties

The beam material is assumed to consist of two ceramics (referred to as ceramic 1 and ceramic 2) and two metals (referred to as metal 1 and metal 2) whose volume fraction varies in both the thickness and longitudinal directions in the form of $V_1(x,z) = V_xV_z$ and $V_2(x,z)$ =*1*-$V_1(x,z)$. Four different material distribution patterns are given as follows [14]:

$$\text{(1a)} \qquad \text{Type I}: \begin{cases} V_x = (x/L)^{n_x}, & 0 \le x \le L \\ V_z = (z/h + 1/2)^{n_z}, & -0.5h \le z \le 0.5h \end{cases};$$

$$\text{(1b)} \qquad \text{Type II}: \begin{cases} V_x = (x/L)^{n_x}, & 0 \le x \le L \\ V_z = \begin{cases} (1+2z/h)^{n_z}, & -0.5h \le z \le 0 \\ (1-2z/h)^{n_z}, & 0 \le z \le 0.5h \end{cases} \end{cases}$$

$$\text{Type III}: \begin{cases} V_x = \begin{cases} (2x/L)^{n_x}\,, & 0 \le x \le 0.5L \\ (2-2x/L)^{n_x}\,, & 0.5L \le x \le L \end{cases} \\ V_z = (z/h+1/2)^{n_z}\,, \quad -0.5h \le z \le 0.5h \end{cases} \tag{1c}$$

$$\text{Type IV}: \begin{cases} V_x = \begin{cases} (2x/L)^{n_x}\,, & 0 \le x \le 0.5L \\ (2-2x/L)^{n_x}\,, & 0.5L \le x \le L \end{cases} \\ V_z = \begin{cases} (1+2z/h)^{n_z}\,, & -0.5h \le z \le 0 \\ (1-2z/h)^{n_z}\,, & 0 \le z \le 0.5h \end{cases} \end{cases} \tag{1d}$$

where n_z and n_x are the grading indexes, which dictate the variation of the constituent materials in the thickness and longitudinal directions, respectively.

P is the material properties (such as the elastic modulus and mass density, etc.) which is determined using the Voigt model as follows [13]:

$$P = V_{c1}P_{c1} + V_{c2}P_{c2} + V_{m1}P_{m1} + V_{m2}P_{m2}\,, \tag{2}$$

where P_c and P_m denote respectively the properties of the ceramic and metal.

2.2 KINEMATICS AND STRAINS

The displacement field, taking into account the shear deformation effect and stretching of thickness, is presented for FG beams as [9]:

$$\begin{aligned} u(x,z) &= u_0(x) - z\frac{\partial w_0}{\partial x} + f(z)\phi_x(x)\,, \\ w(x,z) &= w_0(x) + g(z)\theta(x)\,; \end{aligned} \tag{3}$$

$$\phi_x = \int \theta(x)dx\,.$$

u_0, w_0, θ and ϕ_x are the unknown displacement of the mid-plane of the beam

$$\begin{aligned} & u_0(x) - z\frac{\partial w_0}{\partial x} + k_1 f(z)\int \theta(x)dx\,, \\ & w(x,z) = w_0(x) + g(z)\theta(x)\,, \end{aligned} \tag{4}$$

where

$$f(z) = z\left[1 - \frac{4}{3}\Big(\frac{z}{h}\Big)^2\right]\,, \qquad g(z) = \frac{2}{10}\frac{df(z)}{dz}\,.$$

The kinematic relations can be obtained as follows:

$$\varepsilon_x = \varepsilon_x^0 + zk_x^b + f(z)k_x^s\,, \quad \varepsilon_z = g'(z)\theta\,, \quad \gamma_{xz} = \gamma_{xz}^0(g(z) + k_1 A' f'(z))\,, \tag{5}$$

where

$$\varepsilon_x^0 = \frac{\partial u_0}{\partial x}\,, \quad \varepsilon_z^0 = \theta\,, \quad k_x^b = -\frac{\partial^2 w_0}{\partial x^2}\,, \quad k_x^s = k_1 A' \frac{\partial^2 \theta}{\partial x^2}\,, \quad \gamma_{xz}^0 = \frac{\partial \theta}{\partial x}\,.$$

The integrals used in the above equations shall be resolved by a Navier type method and can be given as follows:

(6) $$\int \theta dx = A' \frac{\partial \theta}{\partial x}\,; \quad A' = -\frac{1}{\lambda^2}\,, \quad k_1 = -\lambda^2\,, \quad \lambda = \frac{m\pi}{L}\,.$$

2.3 Constitutive relations

By assuming that the material of BDFG beam obeys Hooke's law, the stresses in the beam become [9]:

(7) $$\begin{Bmatrix} \sigma_x \\ \sigma_z \\ \tau_{xz} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{11} & 0 \\ 0 & 0 & Q_{44} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_z \\ \gamma_{xz} \end{Bmatrix}\,,$$

where

$$Q_{11} = \frac{E(x,z)}{(1-\nu^2)}\,, \qquad Q_{12} = \frac{\nu E(x,z)}{(1-\nu^2)}\,, \qquad Q_{44} = \frac{E(x,z)}{2(1+\nu)}\,,$$

$(\sigma_x, \sigma_z, \tau_{xz})$ and $(\varepsilon_x, \varepsilon_z, \gamma_{xz})$ are the stress and strain components, respectively.

2.4 Governing equations

Hamilton's principle is used herein to derive the equations of motion. The principle can be presented in analytical form as

(8) $$\int_0^T (\,\delta U - \delta K)\, dt = 0\,,$$

where δU is the variation of strain energyδK and δK is the variation of kinetic energy. The variation of strain energy of the beam can be stated as

(9a) $$\delta U = \int_0^L \int_A [\sigma_x \delta\varepsilon_x + \sigma_z \delta\varepsilon_z + \tau_{xz}\delta\gamma_{xz}] dA\, dx$$

(9b) $$\delta U = \int_A \Big\{ N_x \frac{d\delta u_0}{dx} - M_x^b \frac{d^2 \delta w_0}{dx^2} + k_1 A' M_x^s \frac{d^2 \delta\theta}{dx^2} + N_z \delta\theta + Q \frac{d\delta\theta}{dx} \Big\} dA\,,$$

(9c) $$\begin{Bmatrix} N_x \\ M_x^b \\ M_x^s \end{Bmatrix} = \int_{-h/2}^{h/2} (\sigma_x) \begin{Bmatrix} 1 \\ z \\ f(z) \end{Bmatrix} dz\,;$$

(9d) $$N_z = \int_{-h/2}^{h/2} \sigma_z g'(z) dz \,, \qquad Q = \int_{-h/2}^{h/2} \tau_{xz}(g(z) + k_1 A' f'(z)) dz \,.$$

The stress resultants are obtained in terms of strains as following compact form:

(10a) $$\begin{Bmatrix} N_x \\ M_x^b \\ M_x^s \end{Bmatrix} = \begin{bmatrix} A & B & B^s \\ B & D & D^s \\ B^s & D^s & H^s \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ k_x^b \\ k_x^s \end{Bmatrix} + \begin{Bmatrix} L \\ L^a \\ R \end{Bmatrix} \varepsilon_z^0 \,,$$

(10b) $$N_z = L\varepsilon_x^0 + L^a k_x^b + R k_x^s + R^a \varepsilon_z^0 \,, \qquad Q = A_{44}^s \gamma_{xz}^0 \,,$$

where A, B, D, etc., are the beam stiffness, defined by

(10c) $$\{A, B, B^s, D, D^s, H^s\} = \int_{-h/2}^{h/2} Q_{11}\{1, z, f(z), z^2, z f(z), f(z)^2\} dz \,,$$

(10d) $$R^a = \int_{-h/2}^{h/2} Q_{11}\, g'(z)^2 \, dz \,,$$

(10e) $$\{L, L^a, R\} = \int_{-h/2}^{h/2} Q_{12}\, g'(z)\{1, z, f(z)\} dz \,,$$

(10f) $$A_{44}^s = \int_{-h/2}^{h/2} Q_{44} \left(g(z) + k_1 A' f'(z)\right)^2 dz \,.$$

Also, the variation of the kinetic energy can be expressed as

(11a) $$\delta K = \int_V \rho(x, z)(\dot{u}\delta\dot{u} + \dot{w}\delta\dot{w}) dv \,,$$

(11b) $$\begin{aligned} \delta K = -\int_0^l \Bigg[& I_0(\ddot{u}_0 \delta u_0 + \ddot{w}_0 \delta w_0) + I_1\Big(- \ddot{u}_0 \frac{d\delta w_0}{dx} - \frac{d\ddot{w}_0}{dx}\delta u_0 \Big) \\ & + I_2\Big(\frac{d\ddot{w}_0}{dx}\frac{d\delta w_0}{dx} \Big) + J_0(\ddot{w}_0 \delta\theta + \ddot{\theta}\delta w_0) + J_1\Big(K_1 A' \ddot{u}_0 \frac{d\delta\theta}{dx} \\ & + K_1 A' \frac{d\ddot{\theta}}{dx}\delta u_0 \Big) + J_2\Big(- K_1 A' \frac{d\ddot{w}_0}{dx}\frac{d\delta\theta}{dx} \\ & - K_1 A' \frac{d\ddot{\theta}}{dx}\frac{d\delta w_0}{dx} \Big) + K_0 \ddot{\theta}\delta\theta + K_2 (K_1 A')^2 \frac{d\ddot{\theta}}{dx}\frac{d\delta\theta}{dx} \Bigg] dx \,, \end{aligned}$$

where dot-superscript convention indicates the differentiation with respect to the time variable t; $\rho(x, z)$ is the mass density; and (I_0, I_1, I_2, J_0, J_1, J_2, K_0, K_2) are mass inertias defined as

(11c) $$(I_0, I_1, I_2) = \int_{-h/2}^{h/2} \rho(x, z)(1, z, z^2) dz$$

(11d) $$(J_0, J_1, J_2) = \int_{-h/2}^{h/2} \rho(x,z)(g(z), f(z), zf(z))dz\,,$$

(11e) $$(K_0, K_2) = \int_{-h/2}^{h/2} \rho(x,z)\left(g(z)^2, f(z)^2\right)dz\,.$$

Substituting the expressions for δU and δK into Eq. (8) and integrating by parts and collecting the coefficients of δu_0, δw_0 and $\delta\theta$, the following equations of motion of the beam are obtained:

(12)
$$\begin{aligned}
\delta u_0 :&\quad \frac{\partial N_x}{\partial x} = I_0\ddot{u}_0 - I_1\frac{\partial \ddot{w}_0}{\partial x} + J_1k_1A'\frac{\partial\ddot{\theta}}{\partial x}\\
\delta w_0 :&\quad \frac{\partial^2 M_x^b}{\partial x^2} = I_1\frac{\partial \ddot{u}_0}{\partial x} - I_2\frac{\partial^2 \ddot{w}_0}{\partial x^2} + J_2k_1A'\frac{\partial^2\ddot{\theta}}{\partial x^2} + I_0\ddot{w}_0 + J_0\ddot{\theta}\,,\\
\delta\theta :&\quad -k_1A'\frac{\partial^2 M_x^s}{\partial x^2} - N_z + \frac{\partial Q}{\partial x} = -J_1k_1A'\frac{\partial \ddot{u}_0}{\partial x} + J_2k_1A'\frac{\partial^2 \ddot{w}_0}{\partial x^2}\\
&\quad - K_2(k_1A')^2\frac{\partial^2\ddot{\theta}}{\partial x^2} + J_0\ddot{w}_0 + K_0\ddot{\theta}\,.
\end{aligned}$$

The equation of motion can be expressed in terms of displacements as

(13)
$$\begin{aligned}
\delta u_0 :&\quad \frac{\partial}{\partial x}\left(A\frac{\partial u_0}{\partial x} - B\frac{\partial^2 w_0}{\partial x^2} + B^s k_1A'\frac{\partial^2\theta}{\partial x^2} + L\theta\right) = I_0\ddot{u}_0 - I_1\frac{\partial \ddot{w}_0}{\partial x}\\
&\quad + J_1k_1A'\frac{\partial\ddot{\theta}}{\partial x}\,,\\
\delta w_0 :&\quad \frac{\partial^2}{\partial x^2}\left(B\frac{\partial u_0}{\partial x} - D\frac{\partial^2 w_0}{\partial x^2} + D^s k_1A'\frac{\partial^2\theta}{\partial x^2} + L^a\theta\right) = I_1\frac{\partial \ddot{u}_0}{\partial x} - I_2\frac{\partial^2 \ddot{w}_0}{\partial x^2}\\
&\quad + J_2k_1A'\frac{\partial^2\ddot{\theta}}{\partial x^2} + I_0\ddot{w}_0 + J_0\ddot{\theta}\,,\\
\delta\theta :&\quad -k_1A'\frac{\partial^2}{\partial x^2}\left(B^s\frac{\partial u_0}{\partial x} - D^s\frac{\partial^2 w_0}{\partial x^2} + H^s k_1A'\frac{\partial^2\theta}{\partial x^2} + R\theta\right) - L\frac{\partial u_0}{\partial x}\\
&\quad + L^a\frac{\partial^2 w_0}{\partial x^2} - Rk_1A'\frac{\partial^2\theta}{\partial x^2} - R^a\theta + \frac{\partial}{\partial x}\left(A_{44}^s\gamma_{xz}^0\right) = -J_1k_1A'\frac{\partial \ddot{u}_0}{\partial x}\\
&\quad + J_2k_1A'\frac{\partial^2 \ddot{w}_0}{\partial x^2} - K_2(k_1A')^2\frac{\partial^2\ddot{\theta}}{\partial x^2} + J_0\ddot{w}_0 + K_0\ddot{\theta}\,.
\end{aligned}$$

3 Analytical Solution

For the analytical solution, the Navier method is used for simply supported beams. The solution is assumed to be of the form [11]:

(14a) $$\begin{pmatrix} u_0 \\ w_0 \\ \theta \end{pmatrix} = \sum_{m=1}^{\infty} \begin{pmatrix} U_m e^{i\omega t}\frac{\partial X_m(x)}{\partial x} \\ W_m e^{i\omega t} X_m(x) \\ \theta_m e^{i\omega t} X_m(x) \end{pmatrix}\,,$$

where U_m, W_m and θ_m are arbitrary parameters to be determined, ω is the eigenfrequency associated with (m) the eigenmode. The function $X_m(x)$ for simply supported beam (SS) is given as

$$X_m(x) = \sin(\beta x)\,, \qquad \beta = \frac{m\pi}{L}\,. \tag{14b}$$

Substituting Eqs. (14a) and (14b) into equations of motion, we get the below eigenvalue equations for free vibration problem:

$$([K] - \omega^2[M])\{\Delta\} = \{0\}\,, \tag{15a}$$

where $\{\Delta\}$ denotes the column $\{\Delta\}^T = \{U_m,\, W_m,\, \theta_m$ and

$$[K] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}; \qquad [M] = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}, \tag{15b}$$

with

$$a_{11} = \int_0^L [Ad_{111}X_m + d_1 A d_{11} X_m] d_1 X_m dx\,,$$

$$a_{12} = \int_0^L -[Bd_{111}X_m + d_1 B d_{11} X_m] d_1 X_m dx\,,$$

$$a_{13} = \int_0^L [k_1 A' B^s d_{111} X_m + k_1 A' d_1 B^s d_{11} X_m + L d_1 X_m + d_1 L X_m] d_1 X_m dx\,,$$

$$a_{21} = \int_0^L [Bd_{1111}X_m + 2d_1 B d_{111} X_m + d_{11} B d_{11} X_m] X_m dx\,,$$

$$a_{22} = \int_0^L -[Dd_{1111}X_m + 2d_1 D d_{111} X_m + d_{11} D d_{11} X_m] X_m dx\,,$$

$$a_{23} = \int_0^L \Big[k_1 A'(D^s d_{1111} X_m + 2d_1 D^s d_{111} X_m + d_{11} D^s d_{11} X_m) + d_{11} L^a X_m + 2d_1 L^a d_1 X_m + L^a d_{11} X_m\Big] X_m dx\,,$$

$$a_{31} = \int_0^L [-k_1 A' B^s d_{1111} X_m - 2k_1 A' d_1 B^s d_{111} X_m - k_1 A' d_{11} B^s d_{11} X_m - L d_{11} X_m] X_m dx\,,$$

$$a_{32} = \int_0^L [k_1 A' D^s d_{1111} X_m + 2k_1 A' d_1 D^s d_{111} X_m + k_1 A' d_{11} D^s d_{11} X_m + L^a d_{11} X_m] X_m dx\,,$$

$$a_{33} = \int_0^L \Big[-(k_1A')^2 H^s d_{1111} X_m - 2(k_1A')^2 d_1 H^s d_{111} X_m - (k_1A')^2 d_{11} H^s d_{11} X_m - k_1 A' d_{11} R X_m - 2k_1 A' d_1 R d_1 X_m - 2k_1 A' R d_{11} X_m - R^a X_m + d_1 A_{44}^s d_1 X_m + A_{44}^s d_{11} X_m \Big] X_m dx\,,$$

$$m_{11} = \int_0^L -I_0 d_1 X_m d_1 X_m dx, \qquad m_{21} = \int_0^L -I_1 d_{11} X_m X_m dx,$$

$$m_{12} = \int_0^L I_1 d_1 X_m d_1 X_m dx, \qquad m_{22} = \int_0^L [I_2 d_{11} X_m - I_0 X_m] X_m dx,$$

$$m_{13} = \int_0^L -J_1 k_1 A' d_1 X_m d_1 X_m dx, \quad m_{23} = \int_0^L [-J_2 k_1 A' d_{11} X_m - J_0 X_m] X_m dx,$$

$$m_{31} = \int_0^L J_1 k_1 A' d_{11} X_m X_m dx, \qquad m_{32} = \int_0^L [-J_2 k_1 A' d_{11} X_m - J_0 X_m] X_m dx,$$

$$m_{33} = \int_0^L [K_2 (k_1 A')^2 d_{11} X_m - K_0 X_m] X_m dx.$$

4 Numerical Results and Discussion

The numerical results for the influence of material composition on free vibration of bidirectional functionally graded beam (BDFGB) via a quasi-3D theory are reported and discussed. The beam is composed of Silicon nitride (Si3N4) as ceramic 1, zirconia (ZrO2) as ceramic 2, stainless steel (SUS304) as metal 1 and Titanium (Ti-6Al-4V) as metal 2. The material properties adopted here are the following [13]:

$$E_{c1} = 322.27 \text{ GPa}, \quad \rho_{c1} = 2370 \text{ kg/m}^3, \quad \upsilon_{c1} = 0.3,$$
$$E_{c2} = 116.38 \text{ GPa}, \quad \rho_{c2} = 3657 \text{ kg/m}^3, \quad \upsilon_{c2} = 0.3,$$
$$E_{m1} = 207.79 \text{ GPa}, \quad \rho_{m1} = 8166 \text{ kg/m}^3, \quad \upsilon_{m1} = 0.3,$$
$$E_{m2} = 105.75 \text{ GPa}, \quad \rho_{m2} = 4420 \text{ kg/m}^3, \quad \upsilon_{m2} = 0.3.$$

The non-dimensional fundamental frequency defined as

$$\overline{\omega} = \frac{\omega_1 L^2}{h}\sqrt{\frac{\rho_0}{E_0}}\,, \tag{16}$$

ω_1 is the first fundamental frequency, $E_0 = 70$ GPa, $\rho_0 = 2702$ kg/m^3.

To verify our results, they were compared with literature data. To this end, the fundamental frequencies obtained by the present theory are compared with those obtained by Simsek [21] and Wu [15]. The material properties and geometry of the

2D-FGM beam given in [21] are used.

$$P(x,z) = P_0 e^{n_x(x/l+0.5)+n_z(z/h+0.5)}\,, \tag{17}$$

where P_0 is the reference material parameter, including $E_0 = 210$ GPa, $\rho_0 = 7850$ kg/m^3, and $\upsilon = 0.3$.

The results are tabulated in Table 1. It can be observed that the results are in good correlations with the results obtained by Simsek [21] and Wu [15].

Table 1. Comparison of non-dimensional fundamental frequency of S-S BDFG beam ($L = 20h$)

	Source	$n_z = 0.2$	$n_z = 0.4$	$n_z = 0.6$	$n_z = 0.8$
	Ref. [21]	2.8330	2.8251	2.8115	2.7919
$n_x = 0.2$	Ref. [15]	2.8332	2.8248	2.8109	2.7918
	Present (3D)	2.8359	2.8287	2.8169	2.8004
	Ref. [21]	2.8291	2.8212	2.8076	2.7880
$n_x = 0.4$	Ref. [15]	2.8298	2.8214	2.8076	2.7884
	Present (3D)	2.8359	2.8288	2.8169	2.8004
	Ref. [21]	2.8232	2.8154	2.8017	2.7822
$n_x = 0.6$	Ref. [15]	2.8242	2.8158	2.8020	2.7829
	Present (3D)	2.8360	2.8288	2.8169	2.8005
	Ref. [21]	2.8154	2.8076	2.7939	2.7744
$n_x = 0.8$	Ref. [15]	2.8163	2.8079	2.7941	2.7751
	Present (3D)	2.8360	2.8289	2.8170	2.8005

In the present study we have used a quasi-3D HSDT and the Navier solution for simply supported beams to determine the frequencies. The stretching effect taken into account in the present formulation has an important effect. In this analysis, we considered a constant Poisson's ratio ($\upsilon = 0.3$) because it has negligible effect.

Figures 2 and 3 examine the effect of the gradient indexes (n_x, n_z), aspect ratios (L/h) on the dimensionless fundamental frequency of the BDFG beam, for different type of material distribution. It can be seen from Figure 2 that, for each type of material distribution, the fundamental frequencies increase with the increase of index n_x. In Figure 3, the non-dimensional fundamental frequency decrease with increasing the power law index n_z.

For both figures, with the increase of the aspect ratio, the dimensionless fundamental frequency increase and become very close to each other after $L/h \geq 15$. The reason is an increasing in the grading indexes lead to grow or reduce the beam's stiffness, which increase or decrease the fundamental frequency values.

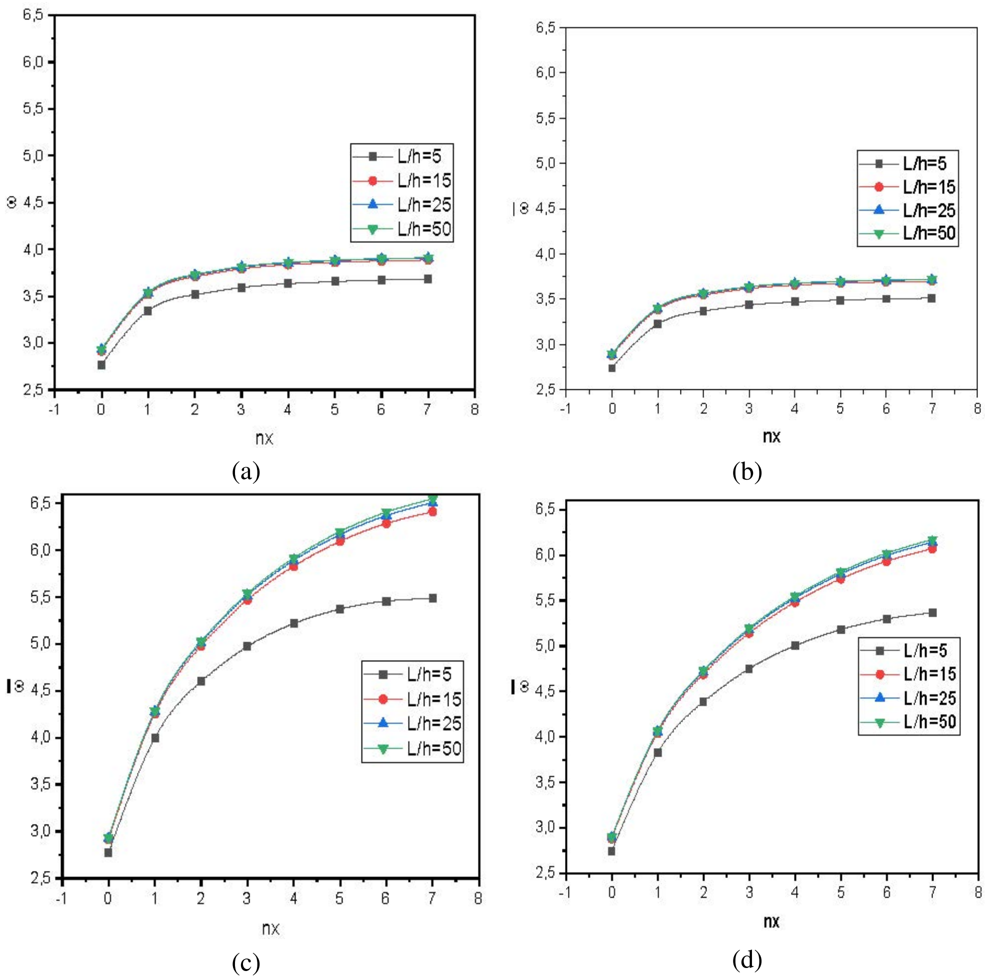

Fig. 2. Variation of the dimensionless fundamental frequency of S-S BDFG beam versus the material gradient parameter n_x for different aspect ratio L/h ($n_z = 1$): (a) Type I; (b) Type II; (c) Type III; (d) Type IV.

Figure 4 shows the effect of the type of material distribution and material gradient parameter n_x on the fundamental frequency of S-S BDFG beam. It can be observed that the highest fundamental frequency values are seen in Type III followed by Type IV, Type I and Type II.

Figure 5 presents the variation of the dimensionless fundamental frequency versus grading index n_z. It can be seen from this figure that the highest dimensionless fundamental frequency values are obtained for Type III. However, the lowest values are predicted for Type II.

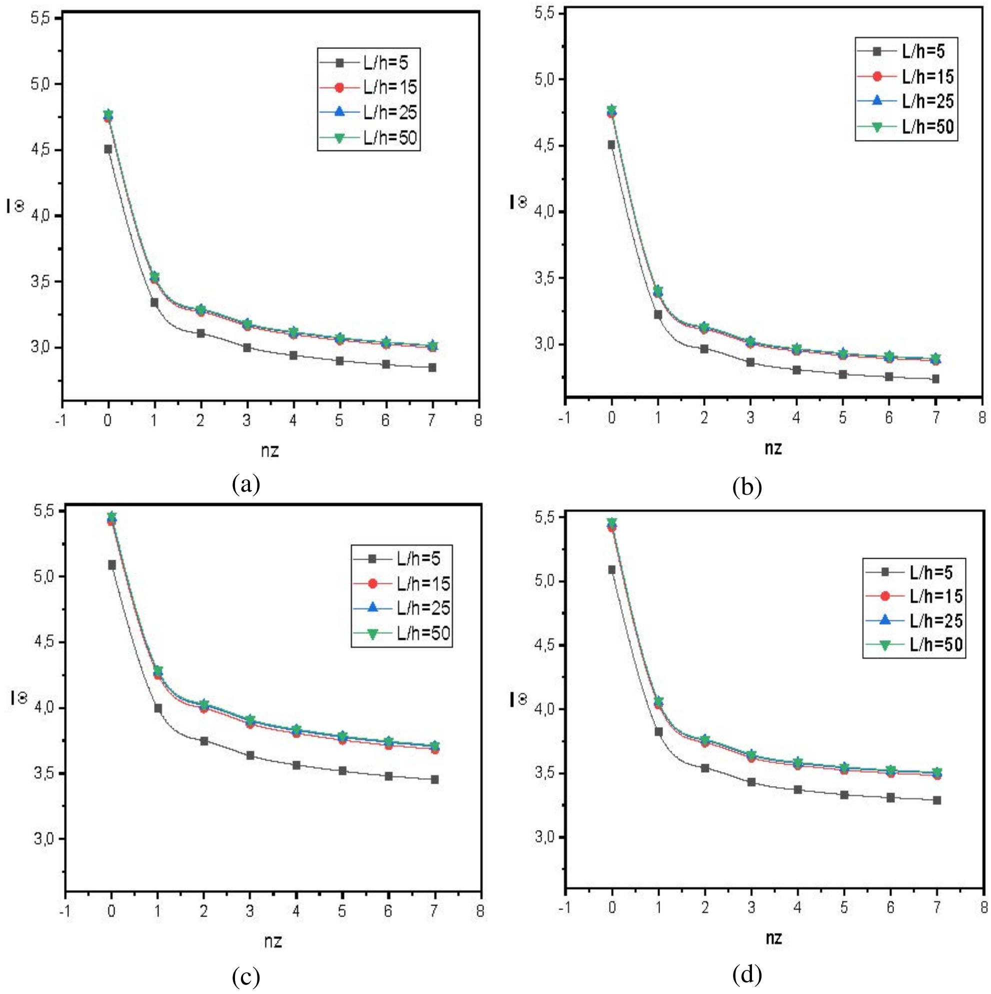

Fig. 3. Variation of the dimensionless fundamental frequency of S-S BDFG beam versus the material gradient parameter n_z for different aspect ratio L/h ($n_x = 1$): (a) Type I; (b) Type II; (c) Type III; (d) Type IV.

Figure 6 examine the variation of dimensionless fundamental frequency with aspect ratio for different type of material distribution. For all type of material distribution, there is a rapid increase in dimensionless fundamental frequency in the region $L/h \leq 15$. Exceeding this value, the curves keep constants.

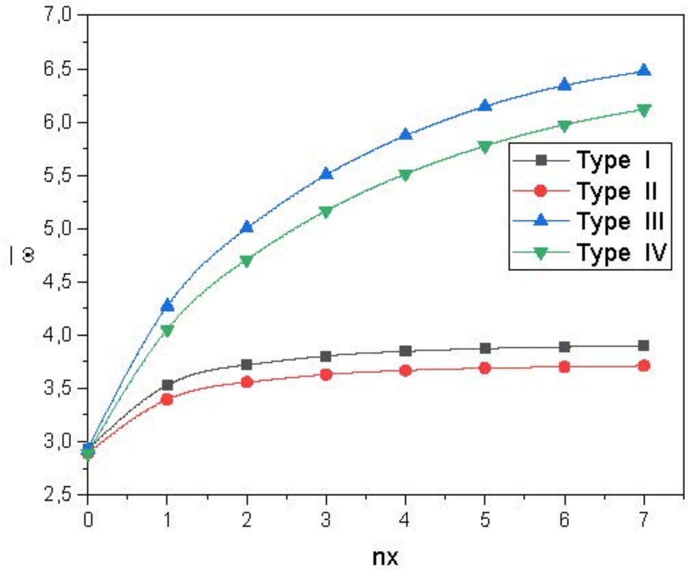

Fig. 4. Effect of the type of material distribution and material gradient parameter n_x on the dimensionless fundamental frequency of S-S BDFG beam ($n_z = 1$, $L/h = 20$).

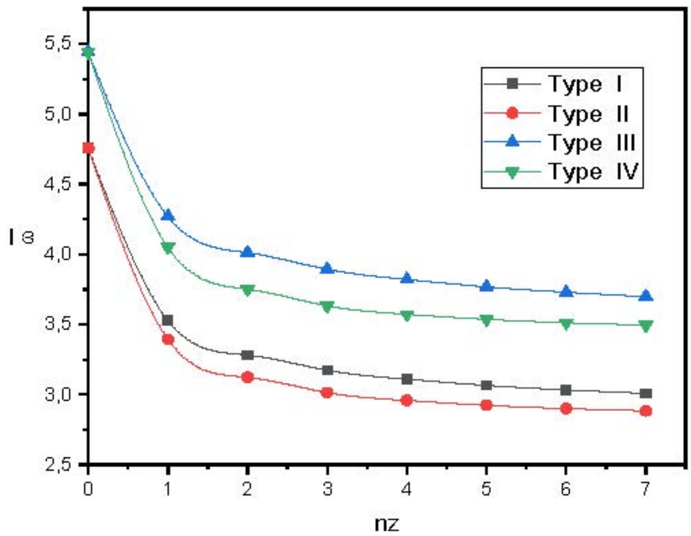

Fig. 5. Effect of the type of material distribution and material gradient parameter n_z on the dimensionless fundamental frequency of S-S BDFG beam ($n_x = 1$, $L/h = 20$).

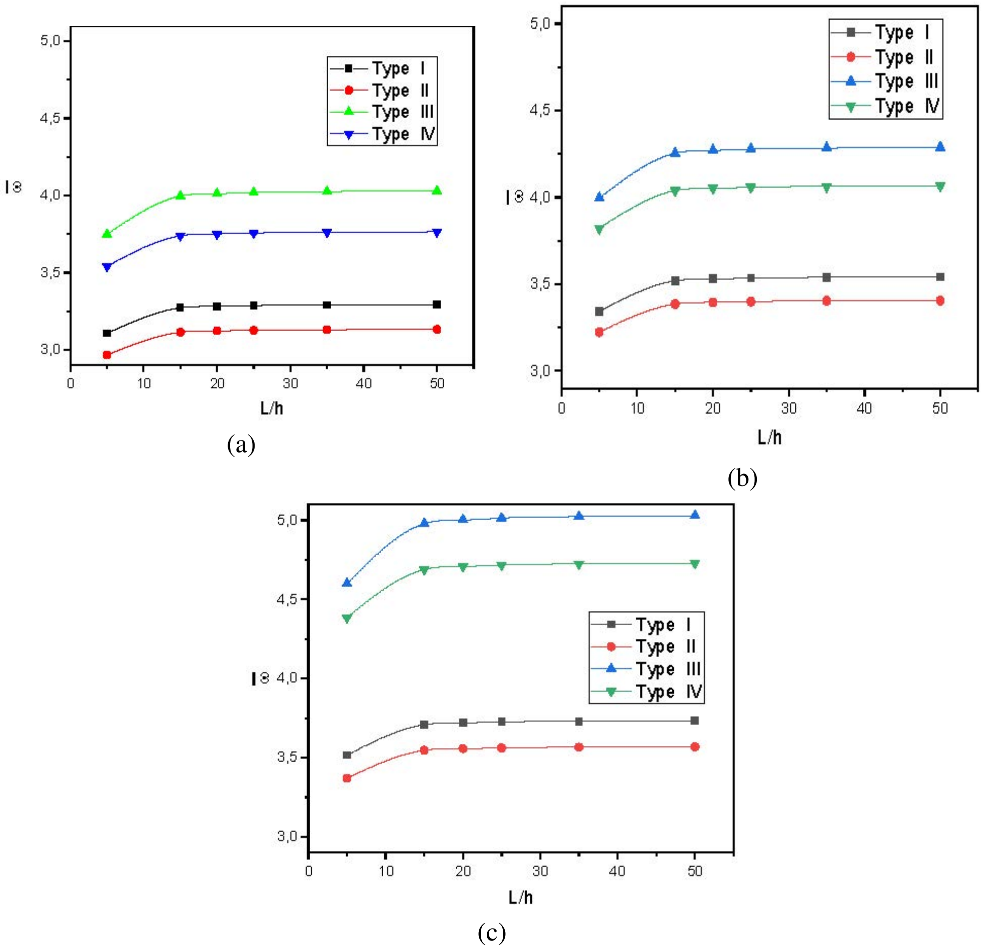

Fig. 6. Effect of the type of material distribution and side-to-thickness ratio L/h on the dimensionless fundamental frequency of S-S BDFG beam: (a) $n_x = 1$, $n_z = 2$; (b) $n_x = 1$, $n_z = 1$; (c) $n_x = 2$, $n_z = 1$.

5 Conclusion

In this paper, material composition on free vibration of bidirectional functionally graded beam via a quasi-3D theory is investigated. The formulation used in this work is based on quasi-3D theory which takes into account the stretching effect. The governing equations are derived from Hamilton's principle and they are solved by Navier solution for simply supported beam. The results show a significant influence of the various types of material distribution patterns on the dimensionless fundamental frequency of the BDFG beam. Indeed:

- The material Type III has the highest dimensionless fundamental frequency. However, the Type II has the lowest ones.
- The aspect ratio L/h have very little influence on the dimensionless fundamental frequencies after the value of $L/h = 15$.
- Increasing the index n_z, which describes the variation of the beam properties in the direction of thickness, leads to a decrease in the frequencies regardless of the index values n_x. Increasing the index values of the latter increases the frequency values independently of n_z.
- The proposed quasi-3D theory is efficient in solving the vibration of the BDFG thick beams.

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