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FLUID MECHANICS

TRANSVERSE STOKES SLIP FLOW PAST AN AXIALLY SYMMETRIC BODIES: A NEW APPROACH

DEEPAK KUMAR SRIVASTAVA*

Department of Mathematics, B.S.N.V. Post Graduate College (University of Lucknow, Lucknow) Station Road, Charbagh, Lucknow-226 001, U.P., India

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ABSTRACT: In the present paper, a closed form solution of transverse Stokes rarefied slip fluid flow past axially symmetric bodies is being considered. The transverse Stokes drag is evaluated for axially symmetric bodies in the slip-flow regime and which is valid for Knudsen numbers, $Kn \leq 0.1$. The extension of Stokes drag on micro-axially symmetric particles from no-slip boundary conditions to slip boundary conditions has been given. It has been concluded in the end that transverse Stokes drag on the micro-axially symmetric particles is equivalent to Stokes solution for continuum flows multiplied by a rarefaction coefficient which is dependent upon the Knudsen number. The author proposed a new approach of providing analytic closed form drag formula for transverse Stokes slip flow past axially symmetric bodies for rarefied gas.

KEY WORDS: Stokes drag, slip flow, micro axially symmetric particle, rarefaction coefficient.

1 INTRODUCTION

From last few years or so, the process of manufacturing and operation of Micro Electro Mechanical Systems (MEMS) or miniaturised devices [1–3] has drawn the attention of physicist, mechanical engineers and fluid mechanists. These MEMS devices are continuously using in instrumentation, micro-rectors, actuators and lab-on-a-chip bio-chemical sensors. The successful operation of these microfluidic systems involve accurate fluid manipulation based on continuum hypothesis or non-continuum hypothesis mainly for fluids as gases. The non-continuum flows were only encountered in low-density (rarefied gas) applications such as vacuum or space-vehicle technology. One of the main difficulties in trying to predict fluid transport in micron-sized devices lies in the fact that the continuum flow assumption implemented in conventional fluid dynamics breaks down because of the very small length scales involved.

For example, the mean free path of air molecules at standard temperature and pressure is approximately 70 nm. Consequently, the ratio of the mean free path of

*Corresponding author e-mail: dksflow@hotmail.com; srivastavadk1971@gmail.com

the molecules to the characteristic dimensions of a MEMS device can be appreciable. This ratio is referred to as the Knudsen number and as it increases, the momentum transfer starts to be affected by the discrete molecular composition of the fluid; in other words, the gas begins to exhibit non-continuum flow effects. One of the main distinctions between conventional rarefied gas dynamics and microfluidic processes lies in the fact that rarefaction effects in MEMS take place at normal operating pressures. The small lengths and fluid velocities employed in MEMS also lead to extremely low Reynolds numbers. Consequently, the analysis of gaseous transport in microfluidic devices often involves the intriguing combination of rarefied gas dynamics and ultra-low Reynolds number flows or creeping flows.

The analytic solutions have been developed for a number of simple device geometries including long straight micro channels. This was George Gabriel Stokes [4] who gave first the analytic form value of drag on sphere in creeping flow under no-slip boundary conditions which is popularly known as "Stokes Law". Barber and Emerson [5] have presented a study describes an extension of Stokes analytical solution for creeping flow past a sphere which takes into account rarefaction (non-continuum) effects. The analysis follows the slip-flow methodology originally proposed by Basset [6]. Datta and Srivastava [7] provided a simple Stokes drag formula based on geometrical variables of axially symmetric bodies in both axial and transverse flow cases under no-slip boundary conditions. In present analysis, author emphasize onto find the analytical expression of drag force experienced by micro-axially symmetric particle subjected to unconfined low Reynolds number fluid flow with the help of simple Stokes drag formula for axially symmetric particles proved in [7] and drag formula for micro-slip-sphere derived in [5]. In other words, the present paper is the generalization of the work of Barber and Emerson from micro sphere to micro axially symmetric particles under the slip flow regime. In other words, author has been extended the expression of Stokes drag on axially symmetric bodies (described in Fig. 1) derived in [7] from Continuum to Non-continuum and presented it in terms of Knudsen number, K_n, with value less than 0.1.

2 Classical Stokes Drag on Axially Symmetric Particle in Newtonian Continuum Fluid

In Fig. 1 geometry of body of revolution of curve is presented in meridional plane x-y (or r-θ, which are polar coordinates) where $P(x, y)$ is any point on curve may be consider in Cartesian coordinates (x, y) or in polar coordinates (r, θ), ψ is the angle between tangent and x-axis, R is the intercepting length between a point on the curve and x-axis, α is the angle between x-axis and intercepting length between a point on the curve varies from 0 to π which is same to the variation of θ in polar coordinates, a_m and b are the maximum length on x-axis and y-axis respectively. U_x and U_y are

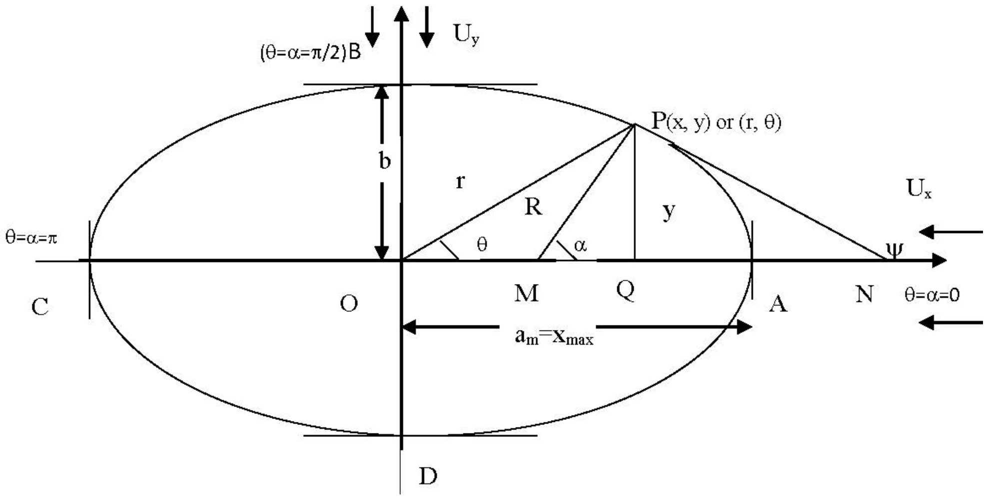

Fig. 1. Geometry of axially symmetric body of revolution of curve in meridional plane.

free uniform fluid stream velocities along axial and transverse directions respectively.

Datta and Srivastava [7] gave the expressions of Stokes drag on axially symmetric particle of revolution of curve in meridional plane with the condition of continuously turning tangent (see Fig. 1) in transverse flow (along y-axis, the transverse flow direction) as

$$F_y = \frac{1}{2}\frac{\lambda b^2}{h_y}\,, \tag{1}$$

where

$$\lambda = 6\pi\mu U_y \tag{2}$$

and

$$h_y = \frac{3}{16}\int_0^{\pi} R\left(2\sin\alpha - \sin^3\alpha\right) d\alpha\,. \tag{3}$$

In the above formulas, variable 'R' is the intercepting length of normal between axis of symmetry and point on body curve in meridional plane and α is the angle made by normal from axis of symmetry. Also, h_y represents the height of centre of gravity of force system in transverse flow configurations and 'b' is the semi-transverse length of body curve in meridional plane xy and 'μ' is the viscosity coefficient [7]. The subscript 'y' indicates that stream is approaching to body along the direction perpendicular to x-axis, i.e. transverse flow is considered. This meridional curve has restrictions of continuously turning tangent avoiding straight line segment, nodes and corners.

In every fluid-particle interaction problem in fluid dynamics, the important quantity to evaluate is the drag experienced by body which in traditional approach always found as a solution of linear steady Stokes equation along with continuity equation

$$\begin{aligned} \mathbf{0} &= -\frac{1}{\rho}\text{grad}p + \nu\nabla^2\mathbf{u}\,, \\ \text{div}\mathbf{u} &= 0\,, \end{aligned} \tag{4}$$

under no-slip boundary condition [8]. In equation (4), ρ is fluid density, p – the pressure, ν – the kinematic viscosity and $\mathbf{u}$ be the fluid velocity.

3 Slip Flow around the Surface of the Axially Symmetric Body

To account for non-continuum flow effects around the microsphere, the Navier-Stokes equations are solved in conjunction with the slip-velocity boundary condition proposed by Basset [6]

$$\tau_t = \beta u_t\,, \tag{5}$$

where u_t is the tangential slip velocity at the wall, τ_t is the tangential shear stress on the wall and β is the slip coefficient. Schaaf and Chambre [9] show that the slip coefficient can be related to the mean free path of the molecules as follows:

$$\beta = \frac{\mu}{\left(\frac{2-\sigma}{\sigma}\right)\lambda'}\,, \tag{6}$$

where μ is the viscosity of the gas, σ is the tangential momentum accommodation coefficient (TMAC) and λ' is the mean free path. For an idealised wall (perfectly smooth at the molecular level), the angles of incidence and reflection of molecules colliding with the wall are identical and therefore the molecules conserve their tangential momentum. This is referred to as *specular reflection* and results in perfect slip at the boundary. Conversely, in the case of an extremely rough wall, the molecules are reflected at a totally random angle and lose their tangential momentum entirely; referred to as *diffusive reflection*. For real walls, some of the molecules will reflect diffusively and some will reflect secularly, and so the tangential momentum accommodation coefficient, σ, is introduced to account for the momentum retained by the reflected molecules. Theoretically, the coefficient lies between 0 and 1 and is defined as the fraction of molecules reflected diffusively. The value of σ depends upon the particular solid and gas involved and also on the surface finish of the wall. TMAC values lying in the range 0.2–1.0 have been determined experimentally [10–12].

Equations (5) and (6) can be combined and rearranged to give

$$u_t = \frac{2-\sigma}{\sigma}\Big(\frac{\lambda'}{\mu}\Big)\tau_t\,, \tag{7}$$

which can be recast in terms of the Knudsen number, Kn as follows:

$$u_t = \frac{2-\sigma}{\sigma}\Big(\frac{Kn\,b}{\mu}\Big)\tau_t\,, \tag{8}$$

where Kn is defined as the ratio of the mean free path of the molecules (λ') to the radius 'b' of the mid-cross section area (Fig. 1)

$$Kn = \frac{\lambda'}{b}\,. \tag{9}$$

The use of Stokes equations with slip boundary conditions on axially symmetric bodies for rarefied gas flows may not fully represent the physical behaviour of the system, especially for moderate to high Knudsen number Kn. Due to this fact and in order to validate the proposed formula, we consider the present study only for low Knudsen number $Kn < 0.1$ for which the approximation of slow flow is valid for rarefied gaseous medium.

4 Proposed drag on Axially Symmetric Body in Stokes Slip Flow

The Stokes drag on axially symmetric body of revolution of meridional curve with 'b' as radius of mid cross-section area (Fig. 1) about x-axis under slip boundary conditions (explained in Section 3) in rarefied medium is given by (replacing λ with λ_1 in equation (1)) [7]

$$(F_y)_{\text{slip}} = \frac{1}{2}\frac{\lambda_1 b^2}{h_y}\,, \tag{10}$$

where

$$\lambda_1 = 6\pi\mu U\frac{1+2\left((2-\sigma)/\sigma\right)Kn}{1+3\left((2-\sigma)/\sigma\right)Kn} = \lambda R_c\,, \tag{11}$$

$$\lambda = 6\pi\mu U\,, \tag{12}$$

$$h_y = \frac{3}{16}\int_0^{\pi} R(2\sin\alpha - \sin^3\alpha)d\alpha \tag{13}$$

and

$$R_c = \frac{1 + 2\left((2-\sigma)/\sigma\right) Kn}{1 + 3\left((2-\sigma)/\sigma\right) Kn}, \tag{14}$$

where R_c is the rarefaction coefficient, dependent on the Knudsen number Kncand σ is the Tangential Momentum Accommodation Coefficient (TMAC).

Author assumes that simple formula of drag (1) on axially symmetric particle placed under transverse flow [7] is also valid and applicable for rarefied gas flow on replacing λ with λ_1 (11) along with low Knudsen number and slip boundary conditions on surface of body.

5 Application of Proposed Formula to Axially Symmetric Bodies

5.1 Slip flow past a sphere

Let us consider the sphere obtained as circle of revolution about x-axis. The parametric equation of circle having radius 'a' as

$$x = a\cos t, \quad y = a\sin t, \qquad 0 \le t \le \pi\,. \tag{15}$$

By detailed derivation, the expressions of transverse Stokes drag on this sphere placed in Newtonian fluid under no-slip boundary conditions, by using (1) and (2) with fact $R = b = a$, $h_x = a/2$, $U_y = U$, comes out to be

$$(F_y)_{\text{no-slip}} = \frac{16\pi\mu U a}{\int_0^{\pi} (2\sin\alpha - \sin^3\alpha) d\alpha} = 6\pi\mu U a\,. \tag{16}$$

Now, the expression of drag on same sphere placed in transverse flow under slip boundary conditions in rarefied medium can be calculated by proposed formula (10) for $R = b = a$, $h_y = a/2$, comes out to be

$$(F_y)_{\text{slip}} = 6\pi\mu U a \frac{1 + 2\left((2-\sigma)/\sigma\right) Kn}{1 + 3\left((2-\sigma)/\sigma\right) Kn} = (6\pi\mu U a) R_c\,, \tag{17}$$

$$(F_y)_{\text{slip}} = R_c\,(F_x)_{\text{no-slip}}\,. \tag{18}$$

The drag coefficient $\dfrac{(F_y)_{\text{slip}}}{6\pi\mu U a} = R_c\,(Kn)$, which confirms with the drag coefficient for microsphere placed under transverse slow slip rarefied gas flow obtained by Beresnev et al. [13] and Stefanov et al. [14].

In the limiting case, as $\lambda' \to 0$, $Kn \to 0$, $R_c \to 1$, provides the classical Stokes drag on sphere [4, 7] exists for Newtonian continuum case i.e., $6\pi\mu U a$.

5.2 SLIP FLOW PAST A SPHEROID

Prolate Spheroid. We considers the prolate spheroid of revolution of ellipse about x-axis. The parametric equation of ellipse (with 'a' as semi major axis length and 'b' as semi minor axis length) is

$$x = a\cos t, \quad y = b\sin t, \qquad 0 \le t \le \pi \,. \tag{19}$$

By detailed derivation, the expressions of h_y and transverse Stokes drag on this prolate spheroid placed in Newtonian fluid under no-slip boundary conditions, by using eq. (1), comes out to be

$$h_y = \frac{3}{32}\frac{b2}{ae^3}\left[\left(3e^2-1\right)L + 2e\right] \,, \tag{20}$$

$$(F_y)_{\text{no-slip}} = 32\pi\mu U ae^3\left[\left(3e^2-1\right)L + 2e\right]^{-1} \,, \tag{21}$$

where $L = \log\dfrac{1+e}{1-e}$ and 'e' is the eccentricity of ellipse.

Now, the expression of Stokes drag on same prolate spheroid placed in transverse flow under slip boundary conditions in rarefied gas medium can be calculated by proposed formula (10), with same value of h_y calculated in (20), comes out to be

$$\begin{aligned}(F_y)_{\text{slip}} &= \frac{1}{2}\frac{\lambda b^2}{h_y} = 32\pi\mu U ae^3\left[\left(3e^2-1\right)L + 2e\right]\frac{1+2\left((2-\sigma)/\sigma\right)Kn}{1+3\left((2-\sigma)/\sigma\right)Kn} \\ &= 32\pi\mu U ae^3\left[\left(3e^2-1\right)L + 2e\right]R_c\end{aligned} \tag{22}$$

$$(F_y)_{\text{slip}} = R_c\,(F_y)_{\text{no-slip}} \,. \tag{23}$$

This expression of drag on micro-prolate spheroid in slip rarefied gas medium in terms of Knudsen number Kn is found to be new and never found in the literature.

In the limiting case, as $\lambda' \to 0$, $Kn \to 0$, $R_c \to 1$, provides the classical Stokes drag on prolate spheroid [7] exists for Newtonian and continuum case i.e.,

$$(F_y)_{\text{no-slip}} = 32\pi\mu U ae^3\left[\left(3e^2-1\right)L + 2e\right]^{-1} \,. \tag{24}$$

Oblate Spheroid. We consider the oblate spheroid of revolution of ellipse about x-axis. The parametric equation of ellipse (with 'a' as semi major axis length and 'b' as semi minor axis length) is

$$x = b\cos t, \quad y = a\sin t, \qquad 0 \le t \le \pi \,. \tag{25}$$

By detailed derivation, the expressions of h_y and transverse Stokes drag on this prolate spheroid placed in Newtonian fluid under no-slip boundary conditions, by using eq. (1), comes out to be

$$h_y = \frac{3}{16}\frac{a}{e^3}\left[\left(1+2e^2\right)\sin^{-1}e - e\sqrt{1-e^2}\right], \tag{26}$$

$$(F_y)_{\text{no-slip}} = 16\pi\mu U a e^3\left[\left(1+2e^2\right)\sin^{-1}e - e\sqrt{1-e^2}\right]^{-1}, \tag{27}$$

where 'e' is the eccentricity of ellipse.

Now, the expression of Stokes drag on same oblate spheroid placed in transverse flow under slip boundary conditions in rarefied gaseous medium can be calculated by proposed formula (10), with same value of h_y calculated in (26), comes out to be

$$\begin{aligned} (F_y)_{\text{slip}} &= \frac{1}{2}\frac{\lambda b^2}{h_y} \\ &= 16\pi\mu U a e^3\left[(1+2e^2)\sin^{-1}e - e\sqrt{1-e^2}\right]\frac{1+2\left((2-\sigma)/\sigma\right)Kn}{1+3\left((2-\sigma)/\sigma\right)Kn} \\ &= 16\pi\mu U a e^3\left[\left(1+2e^2\right)\sin^{-1}e - e\sqrt{1-e^2}\right]R_c \end{aligned} \tag{28}$$

$$(F_y)_{\text{slip}} = R_c\,(F_y)_{\text{no-slip}}\,. \tag{29}$$

This expression of drag on micro-oblate spheroid in slip rarefied gas medium in terms of Knudsen number Kn is found to be new and never found in the literature.

In the limiting case, as $\lambda' \to 0$, $Kn \to 0$, $R_c \to 1$, provides the classical Stokes drag on oblate spheroid [7, 8] exists in Newtonian and continuum case i.e.,

$$(F_y)_{\text{no-slip}} = 8\pi\mu U a e^3\left[e\sqrt{(1+e^2)} - (1-2e^2)\sin^{-1}e\right]^{-1}. \tag{30}$$

6 Conclusion

The closed form expression of transverse Stokes drag has been evaluated for axially symmetric bodies in the slip-flow regime in rarefied medium in terms of Knudsen number with the use of DS-conjecture and the same is valid for Knudsen numbers, $Kn < 0.1$. Rarefaction coefficient R_c depending on Knudsen number Kn separates the two flow features like slip flow in rarefied medium (non-continuum) with no-slip continuum flow of Newtonian fluid. The formula is being applied successfully to micro sphere, micro spheroid (prolate and oblate both). This is the generalization of the work of Barber and Emerson [5] from micro sphere to micro axially symmetric particles under the slip flow regime in rarefied medium. Further, author claims that they have extended the expression of Stokes drag on axially symmetric bodies (described in Fig. 1) derived in [7] from continuum to non-continuum (rarefied medium) and presented it in terms of Knudsen number, Kn. The present work may open the doors to solve many complex fluid mechanical problems in non-continuum mechanics leading from revised DS-conjecture proposed in this paper. Author is exploring these problems and may appear in future papers.

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