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SPECIAL ISSUE

MECHANICAL MODELING OF QUASI-CRYSTALS AT THE MACRO- AND NANO-LEVELS: A STATE-OF-THE-ART REVIEW

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Dedicated to ***Academician Petar Popivanov***

on occasion of his 80th anniversary

ABSTRACT: This work is a state-of-the-art review on the mechanical modeling of macro- and nano-heterogeneous quasi-crystals (QCs)**,** and specifically on elastic, piezoelectric and magneto-electro-elastic (MEE) materials that containing cracks, holes and inclusions. QCs exhibit unique properties, including high stiffness and brittle behavior under both static and dynamic loading, making them promising for advanced technological applications. The paper first summarizes available fundamental solutions and half-space Green's functions in elastostatics and elastodynamics. These provide the foundation for modeling the mechanical response of QCs using advanced computational techniques such as the boundary element method (BEM), meshless methods, Green's function methods and the method of fundamental solutions, all of which serve as a basis for numerical solutions involving various mechanical models. Next, models based on Bak's theory are discussed, highlighting the macro- and nano-scale approaches and including the Gurtin and Murdoch surface elasticity model for nanoscale heterogeneities. The review systematically discusses elastostatic and elastodynamic formulations across different material classes and defect configurations, providing an overview of current modeling capabilities. Finally, open questions and challenges are identified, offering guidance for future research aimed at bridging the gap between theoretical models and practical applications of QCs in high-tech industries.

KEY WORDS: Quasi-crystals; Piezoelectric materials; Magneto-electro-elastic materials; Macro- scale; Nano-scale; Bak's model

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1 INTRODUCTION

The discovery of the first three-dimensional (3D) icosahedral quasicrystal with a five-fold rotational symmetry in a rapidly cooled Al–Mn alloy observed via electron diffraction by Shechtman [1], marked a milestone in material science. The unique atomic arrangement of QCs produces diffraction patterns that cannot be explained using conventional crystallography. Both theoretical and experimental studies have shown that QCs possess a structural complexity far exceeding that of ordinary crystals. The first naturally occurring QCs were discovered in 2009 in a mineral sample from the Koryak Mountains in Russia. Natural QCs are extremely rare, mostly found in certain meteorites and geological formations, with only a few documented occurrences worldwide. Most QCs, including those with piezoelectric and magneto-electro-elastic properties are synthesized in laboratories using rapid solidification, melt spinning [2], or powder metallurgy techniques [3]. Major production and research centers are in Japan, Germany, China and the United States, where synthetic elastic QCs are developed for applications in wear-resistant coatings, aerospace structures and mechanical components. Synthetic QCs dominate practical applications due to controlled production techniques and superior mechanical properties, which allows to exhibit material properties for specific applications. QCs posses an unusual combination of mechanical and physical properties, including high stiffness, low friction, low adhesion, high hardness [4], excellent wear resistance [5], corrosion resistance, low porosity, plus low electrical and thermal conductivities [6]. These attributes enable diverse industrial applications, including protective surface coatings, thermal insulation, solar energy devices, thin films, and reinforcement phases in structural composites [7]. Several applications have been patented [8, 9] and QCs are now used in automotive engineering, energy systems, nuclear fuel containers, aerospace component surface modifications, engine coatings, solar cells, thermoelectric converters and optical technologies. Structurally speaking, QCs are intermetallic solids with quasiperiodic long-range translational order, but lacking conventional crystal periodicity. They are classified as one-, two- or three-dimensional [10]. 1D QCs (Fibonacci chains, Thue–Morse sequences and engineered layered super lattices) are quasiperiodic along one direction and periodic in the perpendicular plane [11]. 2D QCs of octagonal, decagonal and dodecagonal arrangement exhibit quasiperiodicity in two directions, while 3D QCs of icosahedral and cubic arrangement are quasiperiodic along all three spatial directions.

The phenomenological theory of Landau and Lifshitz [12] on the elementary excitation of condensed matter is essentially regarded as the physical basis of QCs elasticity and two types of excitations, *phonons and phasons*, are responsible for quasi-periodicity of materials. The phonons describe conventional elastic deforma-

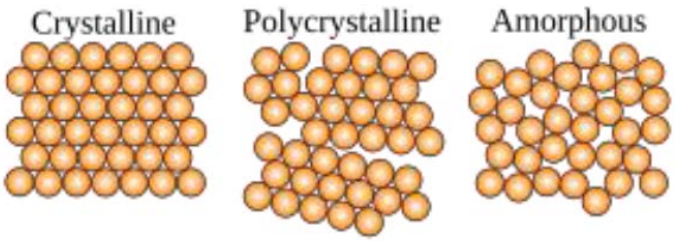

Figure 1. Polycrystalline materials are solids comprising crystals whose size varies from nanometers to millimeters and are separated by grain boundaries with random crystallographic orientation, while amorphous materials do not exhibit long-range order.

tions, while the phasons correspond to local atomic rearrangements that are unique to quasiperiodic structures. The nature of phason dynamics has been debated: Bak [13, 14] proposed that phasons should be treated analogously to phonons, governed by momentum balance equations, while Lubensky et al. [15] argued that phasons exhibit slow diffusive dynamics. Rochal and Lorman [16] introduced a phonon–phason model that unifies both wave-like and diffusive behavior. This theoretical framework has important implications in elastodynamics, noting that Bak's theory serves as a fundamental basis for mechanical models describing QCs fracture. Substantial theoretical and computational efforts have been devoted to predicting QC elastic constants [17, 18]. These includes Monte Carlo simulations [19], relaxation simulations [20] and transfer matrix methods [21]. In terms of experiments, scanning probe microscopy and atomic force microscopy provide powerful tools for nanoscale characterization [22]. 1D QCs have attracted particular attention following the growth of engineered quasiperiodic structures [23], which enable the construction of Fibonacci super lattices and the development of stable 1D QCs [24]. Mechanically, QCs are generally brittle at room temperature and combine high hardness with low fracture toughness. Indentation and tensile tests reveal cleavage fracture with little or no plastic deformation, a behavior comparable to that of ceramics [9,25]. These observations highlight the importance of accurate mechanical models for fracture under both static and dynamic loading.

Comprehensive state-of-the-art reviews on the mechanical modeling of QCs have been generated by Fan et al. [26–29], Fan and Mai [30] for elastostatics, and by Li and Fan [31] for elastodynamics. A review on mathematical elasticity of QCs with defects such as cracks by referring to different boundary-value problems (BVP) in elastostatics is given in [32]. Recent review papers are those of Widom and Mihalkovič [33] predicting the thermodynamic stability of QCs, of Lifshitz [34] dealing with statistical approaches, and of De Donno et al. [35] presenting a mesoscale field theory

unifying the modeling of growth and dislocations in QCs. Although phase-field approaches are beyond the scope of this review, they are briefly discussed as being frequently used the study of QCs. The phase-field framework provides a physics-based description of microstructure evolution by introducing continuous order parameters that minimize a prescribed free-energy functional. Interfaces, quasiperiodic density modulations, and even defect structures arise naturally as solutions to the governing partial differential equations, without the need to track sharp boundaries explicitly [36, 37]. In the context of QCs, phase-field models are particularly powerful, as they can capture both phonon and phason degrees-of-freedom, as well as the competition between multiple length scales that stabilizes quasiperiodic order [35, 38, 39]. These methods are therefore suitable for predicting the formation of pathways, nucleation and growth kinetics, and the evolution of defect networks. By contrast, classical solid mechanics methods are rooted in continuum mechanics, where deformation fields satisfy balance laws of momentum and compatibility, coupled with constitutive relations that generalize Hooke's law to QCs. In such models, phonon and phason fields are treated as coupled displacement fields, and BVP can be solved using well-developed analytical and numerical techniques that are reviewed here. Solid mechanics approaches are thus particularly well suited for analyzing stress concentrations, crack propagation and inclusions, but generally presuppose an existing material microstructure and cannot directly describe structural transitions. In summary, phase-field methods emphasize microstructure formation and evolution from energy principles, whereas solid mechanics methods focus on the stress–strain response of given configurations. The two are therefore complementary: phase-field approaches illuminate how quasicrystalline order forms and transforms, while solid mechanics provides accurate predictions of the mechanical response once the structure has set.

In this review, we focus on results available in the literature concerning the mechanical modeling of QCs within the framework of continuum mechanics. The first ting to note is that there is a limited number of review articles that encompass the broad spectrum of solid mechanics as applied to QCs. The review works discussed above remain seminal references in the field, covering topics such as elasticity, plasticity, defects, and dynamic responses of QCs. To date, Li and Fan [31] and Xu et al. [32] are the most extensive surveys of QC mechanics available in the literature, but do not cover the last few years.

Thus, the aim of this paper is to first summarize available fundamental solutions and half-plane Green's functions in elastostatics and elastodynamics of QCs, considering them as the basis for advanced computational techniques and second to review mechanical models describing the behavior of elastic, piezoelectric and magneto-electro-elastic (MEE) QCs with defects at both macro- and nano-scales under static

and dynamic loading conditions. Briefly, the paper is organized as follows: Section 2 reviews fundamental solutions and half-plane Green's functions, while Sections 3, 4 and 5 discuss elastostatic and elastodynamic models for elastic, piezoelectric and MEE QCs, focusing on the presence of cracks, voids and inclusions. Section 6 concludes the review by summarizing current knowledge and outlining open challenges for future research.

2 Fundamental Solutions and Green's Functions

A generalized Hooke's law for 1D QCs was derived by Wang et al. [11] that provides researchers with a fundamental theory describing their elastic behavior. Numerous investigations on QCs within the framework of linear elasticity, in which a higher-dimensional space and phason field are introduced, were conducted by Fan [27, 29], Ding et al. [40] and Liu et al. [41], while uniqueness and existence of the resulting solutions were investigated in [27]. The relationship of plane elasticity of different point groups (or Laue classes) in 2D QCs was discussed and their equilibrium differential equations in terms of displacements were derived by Liu et al. [42] and the following results were obtained: (i) For dodecagonal QCs, their equilibrium differential equations are simplified as two biharmonic equations; (ii) For pentagonal and decagonal QCs, their equilibrium differential equations are simplified into a fourfold harmonic equation; (iii) For octagonal QCs, only equilibrium differential equations of Laue class 16 are simplified, but the resulting equation is still complicated hindering the solution of this kind of QCs.

The concept of a *fundamental solution* has gradually emerged starting from the latter part of the 19th century and continuing into the early decades of the 20^{th} century. In its earliest form, it was understood as a convolution of distributions used to solve linear partial differential equations with constant coefficients. The first explicit expression for the transient response to a point force in a homogeneous, isotropic and unbounded medium was derived by Stokes [43]. This pioneering result was followed by a series of important contributions from Betti [44], Rayleigh [45], Love [46], Lamb [47, 48], Jaswon [49], Kupradze [50], Cruse and Rizzo [51] and Brebbia [52], thus adding to the development and refinement of the theory. Over time, the availability of fundamental solutions became recognized as a cornerstone for the modern computational techniques such as boundary element method (BEM) (see [53, 54]), of method of fundamental solutions [55, 56], and later for meshless techniques such as the local boundary integral equation method [57]. When we speak of a fundamental solution, what usually comes to mind is the response of an infinite medium to a point load, with the radiation condition imposed because the medium has no boundaries. Once attention shifted to semi-infinite media such as the half-space, boundary conditions emerge and the solutions evolves into *Green's functions*. The next section

reviews fundamental solutions and Green's functions that have been derived for the governing equations describing the mechanical behavior of QCs.

2.1 ELASTOSTATICS

Green's functions play an important role in numerical formulations, because besides their theoretical merits they serve as benchmark solutions to help gauge various numerical methods. Static fundamental solutions for 1D and 2D QCs were derived for elastostatic plane problems in [41, 42] by the complex variable function method. In [58], 3D elastostatic fundamental solutions for 1D QCs of the hexagonal type were obtained, while Gao and Ricoeur [59] derived 3D elastostatic Green's function for the bi-material half-space. Furthermore, they derived fundamental solution for a 2D QCs full-space and pioneered the analysis by Green's functions for cubic QC bi-materials subjected to line forces. 3D elastostatic fundamental solutions for QCs were derived by Wang et al. [60], followed by analytical half-space Green's function on the basis of quasi-harmonic functions [61]. Based on differential operator theory, a fundamental solution for 3D elasticity of the governing equations for 1D hexagonal QCs was obtained in [62] and expressed in terms of four quasi-harmonic functions. The point phonon/phason force solution of an infinite 1D hexagonal QC body was also derived in terms of elementary functions.

Next, the set of partial differential equations satisfied by the phonon and phason static displacement fields in QCs was solved by means of Fourier transform and the eigen-strain method in Ding et al. [63], who discuss in detail the 3D elastic fundamental solutions for different kinds of QCs. The potential function theory of elasticity for 3D icosahedral QCs was developed in [64], where large numbers of complicated elastostatic equations were reduced to a single partial differential equation, and a fundamental solution of the elastostatic equation was expressed by six analytic functions of a complex variable.

Fundamental solutions for 3D static problem of 2D QCs is considered in [65]. Next, general solutions to static elasticity equations for 1D hexagonal QCs and 2D decagonal, hexagonal, dodecagonal and octagonal QCs were studied in [41, 65–69]. Plane elasticity of 3D QCs and general solutions based on stress potential functions was studied in [70], while linear elasticity of cubic QCs was considered in [71–73]. Wu et al. [74] obtained Green's functions for infinite and semi-infinite planes composed of 1D piezoelectric QCs with the aid of the Stroh formalism, while Zhang et al. [75] derived the Green's functions for a 1D piezoelectric QC bi-material by the same method. Building upon this work, Xu et al. [76] further extended the Stroh methodology to 2D piezoelectric QCs and obtained fundamental solution for multi-physics loads. Tables 1 and 2 summarize references that provide fundamental solutions or half-space Green's functions for various types of QCs in 2D and 3D static problems involving elastic and piezoelectric QCs.

Table 1. Fundamental solutions and Green's functions in elastostatics

QCs type	2D Fundamental solution	3D Fundamental solution	2D Half-plane Green's function	3D Half-space Green's function
1D	Liu et al. [41]; Yang et al. [69]	Li [58]; Chen et al. [62]; Wang [77] Gao et al. [66]	Not found	Not found
2D	Liu et al. [42];	Gao and Ricoeur [59] Wang et al. [60]; Ding et al. [63]; Peng and Fan [65]; Gao and Zhao [67,68]	Not found	Gao and Ricoeur [71] Wang et al. [61]
3D	Fan and Guo [64] Gao [72] Li and Fan [70]	Zhou and Fan [73]	Not found	Zhang et al. [78]

Table 2. Fundamental solutions and Green's functions in piezoelectricity for static loads

QCs type	2D Fundamental solution	3D Fundamental solution	2D Half-plane Green's function	3D Half-space Green's function
1D	Wu et al. [74]	Not found	Wu et al. [74] Zhang et al. [75]	Not found
2D	Xu et al. [76]	Not found	Not found	Not found
3D	Not found	Not found	Not found	Not found

2.2 Elastodynamics

The study of elastodynamics of QCs results in additional complications problems for which theoretical physicists have provided various descriptions. Almost all regard the Landau density wave theory as the physical basis of QCs the elasticity [12]. However, a unique opinion on the governing equations for the phason field is missing. According to Bak [13, 14], phasons describe specific structural rearrangements in QCs and can be formulated by analogy to phonons. In this view, phonons and phasons play similar roles in the dynamics, both governed by momentum balance–type equations. By contrast, Lubensky et al. [15] argue that phason field should instead be described by a diffusion equation with a large diffusion time, reflecting slow relaxational dynamics. To reconcile these differing perspectives, Rochal and Lorman [16] proposed a model of coupled phonon–phason dynamics that unifies wave-like and

diffusive-like descriptions, i.e., a framework for different dynamic regimes (wave-like, diffusive-like, or mixed/telegraph-type) depending on the material parameters. In wave-type elastodynamic models based on Bak's theory, phasons propagate like elastic waves and these models are suitable for *high-frequency dynamics* or *transient elastic wave propagation* in QCs where phason inertia is significant. In elasto-hydrodynamic (diffusive) models based on the Lubensky et al. [15] theory, phonons follow wave equations, while phasons are governed by diffusion-type equations and these modes are useful for *low-frequency dynamics*. In wave-telegraph type models, phasons obey telegraph-type equations, which include both wave propagation and damping terms, i.e, phasons propagate but attenuate over time, displaying both *wave-like* and *diffusive behavior*. These last type of models are adequate for intermediate-frequency dynamics, where neither pure wave nor pure diffusion approximations are valid. In this review, focus is on wave-type elastodynamic fracture models based on Bak's theory where phasons propagate as elastic waves. Here, Akmaz and Akinci [79] derived 2D elastodynamic fundamental solutions for 2D QCs by applying the Fourier transform and subsequent matrix transformations to reduce the dynamic plane elasticity problem to a system of uncoupled ordinary differential equations. Thier aim was to finding fundamental solution for QCs with dodecagonal, decagonal and pentagonal symmetry.

A fundamental solution for 3D dynamic problems of 1D hexagonal QCs with Laue class 6/mmm was obtained by Wang [77], while Yaslan [80] considered the same problem but for 1D anisotropic QCs and computed time-dependent fundamental solutions of the phonon–phason displacements and stresses arising from pulse point sources. New existence, uniqueness and stability theorems were also obtained for the dynamic elasticity equations of 1D QCs. Earlier, a 3D time-dependent fundamental solution for 2D QCs was derived by Yakhno and Yaslan [81], where a new method for this derivation involving three space variables in anisotropic 2D QCs was developed. This method works efficiently for 2D QCs with dodecagonal, octagonal, decagonal, pentagonal, hexagonal, trigonal, tetragonal, orthorhombic, monoclinic and triclinic structure of QCs. The same authors [82] worked on a 3D time-dependent fundamental solution for 3D QCs, deriving a series of solutions for graded QCs. In [83], a 3D fundamental solution was derived for an inhomogeneous 2D QCs.

The elastodynamics of graded QCs is discussed in only a few papers [84–86]. A general solutions of the graded, linear elasticity equations for pentagonal QCs was obtained in [84] by minimizing the harmonic elastic energy. In [85], linearly graded strains were considered in decagonal and octagonal QC lattices. Finally, an analytical method of solving the initial value problem for an elastodynamic system of inhomogeneous 3D QCs was developed in [86]. Table 3 illustrates the rather limited number of solutions for elastodynamic fundamental solutions and Green's functions of QCs.

Table 3. Fundamental solutions and Green's functions in elastodynamics

QCs type	2D Fundamental solution	3D Fundamental solution	2D Half-plane Green's function	3D Half-space Green's function
1D	Not found	Wang [77] Yaslan [80]	Not found	Not found
2D	Akmaz and Akinci [79]	Yakhno and Yaslan [81] Altunkaynak [83]	Not found	Not found
3D	Not found	Yakhno and Yaslan [82] Yakhno [86]	Not found	Not found

3 Mechanical Models of Elastic QCs with Cracks, Voids and Inclusions

3.1 Macro-level heterogeneous QCs

Both the static and dynamic behaviour of structural elements such as QCs beams, plates and shells have been studied in [87–95], while defects in QC materials was observed by Zhang and Urban [96]. Experiments have shown QCs to be brittle [97], and as such sensitive to the formation of defects.

Material defects are not only responsible for stress concentration phenomena, but also act also as wave refractors and scatterers. The evaluation of elastic wave field distortions produced by a defect is a process studied in nondestructive testing evaluation of materials and structures, in wave propagation theory with applications in material science and engineering. The defect may be an inclusion, a hole, a crack or lack of smoothness in the boundary geometry. Fracture mechanics concepts can be applied to the continua with defects for assessing the initiation, growth and stability of the defective state and to evaluate the resistance of the material to imposed loads. The following sub-sections review the existing static and dynamic models for QCs.

3.1.1 Elastostatic models for cracked QCs

An extension of fracture mechanics formalism to QCs, otherwise behaving as linear elastic materials, was accomplished by Li et al. [98]. In this work, analytical expressions for the complete stress field in cracked QCs are obtained, while the asymptotic behavior of the stress field around the crack tip indicates a square root singularity. The singularity in the phason stress field arises from that the fact that phonon and phason fields are coupled. A review of the mathematical elasticity theory of QCs with defects such as cracks in reference to different BVPs in elastostatics was given in [32]. According to this review, crack problems involving QCs are rarely presented

in the literature, despite the fact that QCs are brittle. Due to the coupling of phonon and phason displacements in QCs, crack problems are now more complicated than those involving conventional crystals. Fan and Mai [30] further developed linear elastic fracture of QCs and presented a review based on the static elasticity theory. Furthermore, Fan and Guo [64], Zhu and Fan [99] and Li and Fan [100] derived analytic solutions for cracks in 3D icosahedral QCs, while 3D solutions were obtained by Peng and Fan [101, 102] for a crack in infinitely extending 1D hexagonal QCs by using Fourier series and Hankel transform methods.

Crack problems have been addressed in the literature for the following cases:

(i) Geometric configurations of infinite and semi-infinite solids [58, 89, 103–106] and for interfacial cracks [107, 108]

(ii) Line cracks [105, 109]; penny-shaped cracks [89, 104, 110]; planar cracks of arbitrary shape [58, 103, 111, 112]

(iii) Phonon-phason coupling [58, 90, 105, 108]; electro-elastic coupling [103, 106, 113–115].

Results obtained for 2D fracture problems grouped according to anti-plane and in-plane loading conditions are now listed below.

3.1.2 Anti-plane fracture problems

When defects are parallel to the quasi-periodic axis of a 1D hexagonal quasicrystal, the material properties remain unchanged along this direction. The elasticity problem then decomposes into two independent parts: conventional plane elasticity of a hexagonal crystal and an anti-plane phonon–phason coupling problem. The anti-plane case has been studied by Guo et al. [109] for a finite crack in a QC strip using Fourier transforms, thus reducing the BVP to dual integral equations and recovering exact solutions for stresses, strains, displacements and field intensity factors. Specifically, they applied a semi-inverse method in conjunction with Westergaard's stress functions to derive closed-form solutions for a Griffith crack, which reduces to the classical case in the absence of a phason field. Next, using complex variable formalism and conformal mapping techniques, Guo and Liu [116, 117] investigated an elliptic hole with two cracks and determined the mode III stress intensity factors. Based on Stroh-type formalism, Guo and Lu [118] studied four cracks emanating from an elliptical hole under anti-plane shear. Li and Fan [119] obtained exact solutions for two semi-infinite, collinear cracks in a QC strip via a conformal transformation. Yu and Guo [120] analyzed two collinear cracks perpendicular to strip boundaries and used Fourier series expansions to reduce the problem to singular integral equations and thus provide closed-form solutions for stress fields and intensity factors. Finally,

Hu et al. [121] investigated a central crack in a 1D orthorhombic QC strip, and deriving asymptotic expressions of the phonon and phason fields near the crack tips through Fourier transform methods

3.1.3 IN-PLANE FRACTURE PROBLEMS

Mode I and II Griffith cracks have been studied for various QCs, e.g., for 2D octagonal QCs [122, 123], for 2D decagonal QCs [98, 124–126] and for cubic QCs [127–129]. The mathematical approaches used in the solution of the QCs elastostatic fracture problems are Lekhnitskii's formalism [128], the complex variable method [41], Fourier transforms with the singular integral equation method [130], Fourier transforms and stress function theory [124], singular integral equation methods [131], perturbation methods [65], and non-local theory, Fourier transform and the dual integral equation method [132].

More specifically, the 2D crack in 3D QCs subjected to far field loadings was studied in [128] via the generalized Lekhnitskii's formalism, and analytical expressions for both the far field and the asymptotic field near the crack tip were obtained. Liu et al. [41] studied cracks in 1D QCs via the complex variable method, while the 2D cracked decagonal QCs were treated by Guo and Fan [124] using stress functions and Fourier transforms and reducing the coupled phonon–phason elasticity equations to higher-order partial differential equations. Stress and displacement components, as well as stress intensity factors and strain energy release rates, were determined. A Mode-I crack in a 1D orthorhombic QC strip under in-plane phonon and phason stress loading was considered by Hu et al. [130], who used Fourier transforms to reduce the mixed BVP into a system of singular integral equations. Thus, asymptotic expressions of the phonon and phason stresses and displacement fields near the crack tip were obtained in an explicit form. Liu [132] studied the non-local theory solution of a Mode-I crack in a 1D hexagonal QC. Based on the Fourier transform, the BVP is reduced to two pairs of dual integral equations, in which the unknown variables are the displacement jumps across the crack surfaces. Thus, expression of the phonon and phason stress fields at the crack tip were obtained.

A configuration of plane cracks in a 1D orthorhombic QC was analyzed by the singular integral equation method in [131]. Using the relevant fundamental solutions, a set of singular integral equations for a curved crack was derived and two cases, the inclined crack and the arc crack, were discussed. A numerical method was proposed to transform the set of singular integral equations into algebraic equations and upon solution, the stress intensity functions (SIF) were obtained. Numerical results showed that dimensionless SIF values were independent of the material constants, but related to the inclination angle and the curvature of the crack in question. For QC materials, the phonon and phason fields are coupled in the constitutive equations, but the SIF are

not. Only the phonon SIF exists when the phonon load is applied, and similarly for the phason SIF. The results obtained degenerate to the solution of the crack problem in 1D hexagonal and 1D tetragonal QCs. Finally, the perturbation method was used for solving elastic 2D and 3D problems of 2D decagonal QCs with different point groups [65]. They introduced a perturbation parameter in the complicated differential equations governing phonon–phason coupling and succeeded in changing them into a series of uncoupled equations. Thus, SIF for a Mode-I crack in a 2D decagonal QC of point group 10 mm was obtained. Table 4 provides an overview of references addressing the elastostatic fracture problem across different quasicrystal types.

Table 4. Crack problems for QCs in elastostatics

QCs Type	2D anti-plane	2D in-plane	3D
1D	Guo et al. [109] Guo and Liu [116, 117] Guo and Lu [118] Li and Fan [119] Yu and Guo [120] Hu et al. [121]	Liu et al. [41] Hu et al. [130] Liu [132] Sun et al. [131] Zheng et al. [112]	Peng and Fan [101, 102] Zhang et al. [95] Li et al. [133]
2D	Not found	Zhou and Fan [122] Yin et al. [123] Li et al. [98] Guo and Fan [124] Li and Fan [125] Yu [126] Peng and Fan [65]	Peng and Fan [65] Li et al. [110]
3D	Li and Chai [134]	Gao et al. [127–129]	Not found

3.1.4 Elastostatic models for QCs with voids and inclusions

By employing conformal mapping and the complex variable method, Fan [27] obtained exact solutions for elliptical holes in 1D hexagonal QCs. Static elastic problems involving inclusions in QCs were investigated by Wang [135] and Gao and Ricoeur [136]. It has been shown that using QCs as inclusion materials can effectively hinder dislocation motion, thereby enhancing the toughness and strength of the surrounding matrix [137, 138]. Wang et al. [139] derived a first-order approximation to the analytical solution for the displacements in both phonon and phason fields, which were induced by a spherical inclusion in icosahedral QCs. Further studies regarding circular, spherical, and ellipsoidal inclusions in 1D and 2D QCs were conducted by Gao et al. [127], Zhang et al. [140], Zhai et al. [141] and Li et al. [142, 143]. Lou

et al. [144] examined a thin elastic inclusion embedded in an infinite 1D hexagonal QC and developed a numerical approach to solve the associated hypersingular integral equation formulation using the finite-part integral method. In [146], ideal elastic behavior that neglects the dissipation associated with the atomic rearrangements is considered. Under these conditions, the decagonal QC plate bending problems have been investigated using Stroh formalism and analytical solutions were provided for a decagonal QC plate with an elliptic hole subjected to out-of-plane bending moments.

3.1.5 Elastodynamic models for cracked QCs

A review on the elastodynamics of QCs is given in [31,147] that included the in-plane and anti-plane cases. The different viewpoints on the role of the phason variables in the dynamic processes are discussed and summarized follows: (a) The phason field is similar to the phonon field in QCs, with both representing long-wavelength propagation; (b) The phason field is diffusive, so it plays a different role than phonons in the hydrodynamics of QCs. Though some researchers believe that the models based on Lubensky et al. [15] presents the most fundamental viewpoint, this theory has not been verified experimentally.

Elastodynamic problems based on the Bak [13, 14] viewpoint are next discussed. The following basic equations for QCs were formulated by Li and Fan [31]: (a) anti-plane elastodynamics for 1D hexagonal, 3D icosahedral and 3D cubic QCs; (b) in-plane problems for dodecagonal QCs; (c) approximate forms of 2D elastodynamics for 3D icosahedral QCs. Finally, a finite difference scheme was developed for 2D dynamic problems involving a centred rectangular cracked specimen made of 2D decagonal Al-Ni-Co and 3D icosahedral Al-Pd-Mn QCs under transient dynamic loads. Next. a centred in-plane crack in a finite, homogeneous or functionally graded Al-Ni-Co quasicrystal solid under suddenly applied and maintained transient loads was considered in [148, 149] using the meshless local Petrov–Galerkin method. The scattering of Love waves by an interface crack between a 1D hexagonal QC coating and a half-space elastic substrate was also investigated by using the integral transform technique. The wave scattering problem was transformed into Cauchy singular integral equations, followed by solution for the dynamic stress intensity factors (DSIF) at the crack tips. This was followed by parametric studies on the effect of the material composition, incidence wave direction, crack sizes and the coupling coefficient on the DSIF. Results showed that appropriate material combinations and incidence wave directions can hinder crack expansion. Moreover, the smaller the absolute value of the coupling coefficient and of the crack size, the smaller the peak value and the volatility of DSIF.

Wang and Schiavone [150] presented the Stroh formalism for 2D subsonic steady-state motion of anisotropic QCs. Ma et al. [151–153] explored wave scattering prob-

lems for an interface crack between a 1D hexagonal QCs coating and an elastic substrate under harmonic plane waves. The Fourier transform method was used to obtain Cauchy singular integral equations of the second kind, in which the unknown variables are dislocation density functions. By taking the limit of the stress field near the crack tip, DSIF and crack opening displacements (COD) were obtained. Finally, numerical results for DSIF resulting from various elastic waves, material combinations, crack sizes, incident angles, and coupling coefficients in the phonon-phason field. Zhao and Li [154] investigated the scattering problem of SH waves by a crack in 1D hexagonal QCs, while in [28] the fracture theory of QCs is presented covering linear, nonlinear and dynamic fracture problems for different quasicrystal systems, with attention paid to 3D The dynamic behavior of interface cracks in a PQC icosahedral and 2D decagonal QCs. Table 5 summarizes the references on the elastodynamics of macro crack problems for QCs.

Table 5. Elastodynamic macro crack problems for elastic QCs

QCs Type	2D anti-plane	2D in-plane	3D
1D	Ma et al. [151] Zhao and Li [154]	Ma et al. [152] Ma et al. [153]	Not found
2D	Li and Fan [31]	Sládek et al. [148, 149]	Not found
3D	Li and Fan [31]	Li and Fan [31]	Not found

3.1.6 Elastodynamic models for QCs with holes and inclusions

To the authors' best knowledge there are no results concerning wave scattering by holes and inclusions in QCs based on the Bak theory [13, 14].

3.2 Nano-heterogeneous QCs incorporating the Gurtin and Murdoch surface elasticity model

The mechanical behavior of materials at the nanoscale has recently attracted considerable attention in the field of mechanics. Nanomechanics extends the concepts and methods of traditional solid mechanics to describe mechanical response at the nanoscale. Advanced continuum-based models in nanomechanics broaden the scope of classical continuum mechanics by integrating fundamental theoretical principles with the distinct effects observed at the nanoscale. Among such models are nonlocal elasticity theory [155], higher-order strain-gradient and higher-order non-local strain-gradient elasticity theories [156], as well as surface elasticity models, with the Gurtin and Murdoch [157] formulation being the most widely used. Specifically, Gurtin and Murdoch [157] pioneered a general theoretical framework to account for

both surface and interface stresses and their influence on the overall stress–strain behavior of nanocomposites. In their model, the interfaces between nanoscale heterogeneities and the surrounding elastic matrix are treated as thin membranes with their own mechanical properties and surface tension. Furthermore, it is assumed that residual stresses arise at surfaces and interfaces even in the absence of external loading. These residual stresses are size-dependent, and their impact becomes particularly significant for nanosized objects with a high surface-to-volume ratio. Surface elasticity effects arise because molecules located near surfaces and interfaces occupy equilibrium positions and energy states that differ from those in the bulk material. For structures larger than approximately 100 nm, the surface-to-volume ratio is negligible, and the effective material properties are governed primarily by the classical bulk elastic strain energy, expressed through a fourth-order elastic stiffness tensor. At smaller length scales, however, surface effects can substantially modify the strain energy, leading to both local and macroscopic material properties that deviate sufficiently from bulk behavior to provide the foundation for nanotechnology. Application of this model to elastodynamic wave propagation in homogeneous and inhomogeneous elastic, piezoelectric, and magneto-electro-elastic nano-heterogeneous solids composed of classical crystals has been addressed the authors' publications [158–162]. Building on these results, the present study turns to nano-heterogeneous QCs and reviews the existing literature on analysis within the framework of the Gurtin and Murdoch model [157].

3.2.1 ELASTOSTATICS

Classical elastic theory of QCs does not admit size-dependent phenomena since there is no intrinsic length scale involved in the constitutive laws. The surface elasticity theory proposed by Gurtin and Murdoch [157] has predicted size-dependent elastic properties that are in good agreement with certain experimental results or atomic calculations, and which have been widely used to study elastic behavior of solids at nano-scales [163, 164]. A number of nanoscale icosahedral QCs phase formations have been reported in Zr-based alloys [165]. However, an analysis of the mechanical properties of a nanosized quasicrystal material has not been discussed sufficiently with consideration to surface elasticity. Therefore, it is necessary to develop a size-dependent elastic theory of a QC, which accounts for the microstructure of the material. In [166], the classical surface elasticity theory of Gurtin and Murdoch was indeed developed for a QC material. As an illustrative example, the effect of surface elasticity on the decagonal QC material weakened by a nanoscale elliptical hole was examined and explicit expressions for the hoop stress around the edge of the hole were obtained. Recent advances in the elastostatics of nano-cracks in QCs emphasize the importance of size-dependent continuum theories. Sládek et al. [167,168] ex-

tended gradient elasticity to QCs, deriving a path-independent J-integral and showing that nanoscale effects regularize classical crack-tip singularities. Wang and Fan [169] and Wang and Schiavone [170] employed the Gurtin and Murdoch surface elasticity model to obtain analytical solutions for nano-cracks in finite quasicrystal domains and their interaction with dislocations, highlighting strong surface-induced modifications of the stress fields. More recently, Li et al. [143, 145] introduced a phase-field approach for brittle fracture in 2D QCs and provided a computational framework capable of capturing mixed-mode nano-crack evolution and phonon–phason coupling. Lian-He and Guo-Hong [171] evaluate the effect of the residual surface stress on a decagonal QC weakened by a nanoscale elliptical hole based on a variation principle, a continuum theory of surface elasticity and new boundary conditions for QCs. The explicit expressions for the hoop stress along the edge of the hole are obtained using the Stroh formalism. The results show that residual surface stress and the shape of the hole have a significant effect on the elastic stress field around the hole. All these studies enhance the understanding of nano-crack behavior in QCs, highlighting the importance of incorporating size-dependent effects and surface elasticity in modeling and analysis.

3.2.2 Elastodynamics

Dynamic loading by elastic waves plays a crucial role in nanostructure mechanics, by redistributing the displacement and stress fields and generating localized concentrations around nano-defects that can lead to material failure. However, most studies focus on macro-defect models, neglecting size-dependent effects and nanoscale heterogeneity, leading to a poor understanding of the behavior of nano-heterogeneous QCs under dynamic loads.

4 Mechanical Models of Piezoelectric QCs with Cracks, Holes and Inclusions

Fully realized piezoelectric quasicrystals (PQCs) have not yet been confirmed experimentally in bulk. Theoretically, QCs with certain symmetries, particularly 1D hexagonal and 2D decagonal configurations lacking inversion symmetry, have been predicted to exhibit piezoelectric coupling, mediated not only by the phonon fields but also by unique phason field contributions to polarization. From an experimental perspective, direct detection of piezoelectricity in QCs remains elusive. While related electromechanical responses, such as anomalous dielectric behavior and phason-assisted polarization effects, have been reported in decagonal Al–Ni–Co and Al–Cu–Fe systems, reproducible bulk piezoelectric measurements are lacking. Current research is therefore concentrated on two main directions: (a) *alloy engineering of quasicrystalline phases* with non-centrosymmetric space groups and (b) *thin-film*

quasicrystalline composites or *metamaterial architecture*, where inversion symmetry breaking may be engineered at the nanoscale. Thus, while the theoretical framework for piezoelectric quasicrystals is well established, experimental realization as functional piezoelectric materials remains an open challenge. As a result, numerous constitutive models and analytical studies have been developed to capture this effect, addressing both static and dynamic problems for PQCs with cracks, inclusions, and surface effects.

4.1 Macro-level heterogeneous PQCs

4.1.1 Static load conditions

Piezoelectricity originates from the breaking of inversion symmetry in crystal structures. In materials without inversion symmetry, application of mechanical stress induces the accumulation of electric charges, giving rise to the piezoelectric effect. QCs have been shown to exhibit this property, as evidenced by the determination of independent, non-vanishing first-order piezoelectric, piezomagnetic, pyromagnetic, photoelastic, and magnetoelectric coefficients [172–177]. The piezoelectric properties of QCs promise a broad spectrum of applications, including sensors and actuators, high-precision acoustic wave devices, and energy-harvesting systems capable of converting mechanical vibrations into electrical energy. Their integration into micro- and nano-scale architectures further expands their potential for signal detection and control. Moreover, the combination of structural stability and piezoelectricity in QCs opens pathways for developing sensing technologies capable of operating reliably in harsh or extreme environments. The following section addresses recent results on the mechanical behavior of PQCs with macro-cracks and inclusions under static loading conditions. Research on the fracture behavior of PQCs has developed along several complementary directions, ranging from the derivation of governing equations and general solutions for the analysis of cracks, cavities, and inclusions, as well as the evaluation of effective electro-elastic properties.

By applying a variational principle, Altay and Dokmeci [178] derived the 3D governing elasticity equations for QCs incorporating piezoelectric effects. Utilizing the Stroh formalism and conformal mapping, Green's functions of generalized displacements and stresses are obtained in [179] for two general cases: a semi-infinite crack and a finite crack in PQCs. Along the same lines, Zhang *et* al. [180], used operator theory and the generalized Almansi theorem to obtained general solutions to plane problems for 1D orthorhombic PQCs. Shortly thereafter, Li et al. [181,182] extended these results by deriving 3D static solutions for 1D hexagonal PQCs and establishing Green's functions for both infinite and half-infinite spaces subjected to external thermal loading. Yu et al. [183] employed both the operator method and complex

variable formalism to obtain general solutions for piezoelasticity problems in 1D QCs with monoclinic, orthorhombic, tetragonal and hexagonal symmetries. Their analysis yielded closed-form solutions for the phonon, phason and electric fields in the vicinity of an anti-plane crack tip. Fan et al. [103] investigated 3D cracks in 1D hexagonal PQCs and by using the Hankel transform, derived fundamental solutions for unit-valued, point extended displacement discontinuities, including phonon and phason displacement discontinuities as well as electric potential discontinuities. Based on these results, they formulated boundary integral equations for planar cracks of arbitrary shape in a 1D hexagonal PQCs periodic plane, which was expressed in terms of extended displacement discontinuities. Using the boundary integral equation method, they investigated the singular behavior of near-crack fields and derived the corresponding stress and electric displacement intensity factors. Pi et al. [184] also used complex variable formalism to analyze mode III cracks in 3D cubic PQCs. A Hamiltonian-based isogeometric analysis was introduced by Yang et al. [185] to analyze mode III fracture in 1D hexagonal PQCs and parallel progress was made in the study of the anti-plane cavity problem. Yu et al. [186] analyzed an elliptical cavity in 1D hexagonal PQCs under anti-plane conditions using Stroh-type formalism. Expanding on this line of inquiry, Yang and Li [206] investigated the anti-plane shear problem for a circular hole containing a straight crack in 1D hexagonal PQCs. This was followed by Zhou and Li [187, 188], who employed the integral equation approach to obtain exact solutions for two symmetrically distributed, collinear mode-III cracks, oriented both parallel and perpendicular to the surface of 1D hexagonal PQCs. Zhang et al. [189] solved the Mode-I penny-shaped crack problem in an infinite 1D hexagonal PQC. More specifically, analytical solutions of the resulting integro-differential equations were obtained for both electrically impermeable and permeable crack surfaces. Using the generalized potential theory method, complete 3D solutions under symmetric concentrated and uniform loads were expressed in terms of elementary functions, and numerical examples demonstrated the coupling effects of phonon, phason and electric fields. Guo and Li [198] investigated the anti-plane problem for 1D hexagonal PQCs with an elliptical inclusion, while Hu et al. [190, 191] analyzed the electro-elastic fields in circular and cylindrical inclusions for 1D PQCs, deriving SIF and electric displacement distributions for both phonon and phason fields.

Building on conformal mapping techniques, Zhang et al. [192] considered the interaction problem between a screw dislocation and an elliptical inclusion in 1D hexagonal PQCs, and provided the corresponding analytical solutions. In addition, Zhou et al. [193] studied the T-stress of a Griffith crack in 1D hexagonal PQCs, deriving a closed-form solution and presenting numerical results that revealed the influence of both electric and phason fields on the T-stress distribution. Li et al. [143, 145]

analyzed the effective electro-elastic properties of 1D hexagonal PQCs containing randomly oriented inclusions, and demonstrated that both the orientation and aspect ratio of the inclusions exert a significant influence on the response of the material. In summary, these studies provide a solid theoretical framework for PQC electro-elastic behavior under various loading and defect conditions. Despite progress in deriving analytical solutions, challenges remain when addressing complex geometries, dynamic and multi-physical couplings and irregular defects.

4.1.2 Dynamic load conditions

Altay and Dokmeci [178] investigated the general 3D governing equations of homogeneous PQCs formulated in the time domain, and in both differential and variationally invariant forms. Building on this foundation, Yang et al. [194, 195] examined dynamic crack problems in functionally graded 1D hexagonal PQCs subjected to horizontally polarized shear (SH) waves. In [194], the dynamic interaction between an SH-wave and a crack in graded PQCs was analyzed using integral transform techniques. The material properties were assumed to vary exponentially in the direction perpendicular to the crack face. By applying the Fourier transform, the problem was reduced to three pairs of dual integral equations, which were solved numerically via the Copson method. Explicit expressions for the phonon, phason, and electric fields on the crack surface were derived, along with DSIFs at the crack tip. Numerical examples highlighted the effects of wave amplitude, incidence angle, wave number, crack length, and gradient parameters on the normalized DSIFs. Using a similar approach, Yang et al. [195] extended the analysis to the dynamic fracture of a partially permeable crack in functionally graded 1D hexagonal PQCs under time-harmonic SH waves. The dynamic behavior of interface cracks in a PQC composite systems has also been studied, e.g., by Ma et al. [152] who analyzed an interface crack in a 1D hexagonal PQC coating bonded to a semi-infinite homogeneous elastic substrate subjected to time-harmonic elastic waves. By introducing dislocation density functions, the scattering problem of plane waves was reduced to a set of Cauchy-type singular integral equations of the second kind. Their results showed that the presence of the phason field amplifies DSIFs fluctuations at crack tips, and that strong phonon–phason coupling promotes crack propagation. Finally, Panja and Mandal [196] investigated the interaction of anti-plane shear waves with two collinear cracks in 1D hexagonal PQCs. Using Fourier transform techniques, the mixed BVP was reduced to three pairs of dual integral equations and solved analytically via the Hilbert transform. Closed-form expressions were obtained for DSIFs of the phonon and phason fields, crack opening displacements, and electric displacements. In conclusion, despite progress, analytical and numerical tools for dynamic fracture in composite PQCs remain limited, particularly for wave interactions with defects and the cou-

pled phonon–phason–electric fields. Advancing the field requires efficient numerical methods for complex geometries, nonlinear behavior and multiphysics interactions, combined with experimental validation, so to unlock the potential of PQCs for modern engineering applications

4.2 NANO-LEVEL HETEROGENEOUS PQCS

4.2.1 STATIC LOAD CONDITIONS

Few studies have addressed defects in nano-scale PQCs so far. Building on the surface/interface model [157], several investigators have analyzed defects in piezoelectric materials [197–199]. Specifically, these researchers investigated the anti-plane shear problem for an elliptic nano-hole and a nano-crack bonded to a reinforcing matrix in 1D hexagonal PQCs under remote mechanical and electrical loading. Using conformal mapping, they derived exact expressions for phonon and phason stresses and electric displacements in the surrounding matrix under both electrically permeable and impermeable boundary conditions. As the elliptic nano-cavity degenerates into a crack, three size-dependent field intensity factors near the crack tip were obtained.

Ma et al. [200] modeled the anti-plane problem for a sharp nano-crack in 1D hexagonal PQCs with electrically semi-permeable conditions. Analytic solutions for the electroelastic field were obtained by conformal mapping and potential theory. Results concerning hole-crack interaction at the nano-scale were obtained by a number of researchers [115, 201–205], all within the framework of the Gurtin and Murdoch model. Specifically, Yang and Liu [115] studied the anti-plane fracture problem involving three nano-cracks emanating from a hexagonal nanohole in 1D hexagonal PQCs using the complex-valued potential theory. Exact solutions for SIF of the phonon and phason fields, as well as for the electric displacement intensity factor, were derived under both permeable and impermeable boundary conditions. Numerical examples demonstrated the significant influence of surface effects on these intensity factors, showing that the coupling of phonon, phason, and electric fields becomes more pronounced as the cavity size decreases.

Other related contributions include (a) Su et al. [201], who studied the mode-III fracture problem of an elliptical nano-hole with double collinear cracks in 1D hexagonal PQCs; (b) Su and Xiao [202], who analyzed the mode-III fracture problem of four nano-cracks emanating from a nanoscale elliptical hole and obtained analytical solutions for electro-elastic field intensity factors and energy release rates. Their work examined the influence of defect size, multi-crack interaction, phonon–phason coupling, and type of applied loads on fracture behavior; (c) Xin and Xiao [203], who investigating a through-crack at the edge of a circular nano-hole. They derived analytic

expressions for phonon, phason, and electric fields, as well as field intensity factors at both crack tips. Their solution degenerates to known results as special cases, and highlights the strong dependency of the field intensity factors on defect size, crack location, far-field loading and coupling coefficients; (d) Wu et al. [204] constructed a conformal mapping from a regular 2n-polygon hole with two asymmetric collinear cracks into a circle. Using complex potential theory, they studied electrically semi-permeable boundary conditions in 1D hexagonal PQCs and derived size-dependent stress intensity factors for phonon and phason fields, as well as electric displacement field intensity factors at the crack tips. (e) Wang et al. [205] employed the surface elasticity model in conjunction with novel conformal mapping technique to address the Mode III fracture of a nano-lip-shaped hole with four emanating cracks in 1D hexagonal PQCs. Closed-form expressions for the SIF were derived, from which solutions for several classical defect configurations can be obtained through parameter degeneration. Numerical simulations were subsequently conducted to investigate the influence of hole's size, crack length, external mechanical and electrical loading, and the phonon–phason coupling coefficient on the fracture behavior.

All the above studies highlight the critical role of surface and size effects in nanoscale PQC fracture. Strong coupling among phonon, phason, and electric fields significantly affects stress distribution and fracture behavior, especially as defect size decreases. Analytical and numerical solutions using surface/interface models, complex variable methods and conformal mapping offer valuable insights, yet the field remains underexplored, particularly for complex defect geometries, multi-defect interactions, and dynamic or multi-physics loading, indicating opportunities for future research.

4.2.2 Dynamic load conditions

To the best of the reviewer's knowledge, no studies have comprehensively addressed the dynamic fracture of nanoscale cracks and inclusions in PQCs, see Table 6. This gap is significant, as dynamic loading is inherent to many PQC applications, e.g., sensors, actuators, and energy harvesters, and nanoscale defects can strongly affect fracture initiation and propagation. Bridging this gap requires advanced theories and computational models to capture the combined dynamic loading environment, the nanoscale surface elasticity, and the unique electromechanical coupling of PQCs.

5 Mechanical Models of Magneto-electro-elastic QCs with Cracks, Holes and Inclusions

We now return to the state-of-the art regarding mechanical models for macro- and nano- heterogeneous, magneto-electro-elastic quasicrystals (MEE QCs). These materials combine aperiodic atomic structure with intrinsic magneto-electric coupling,

Table 6. Fracture in PQCs and in MEE QCs

Problem type and defect scale	PQCs: Nanoscale	PQCs: Macroscale	MEE QCs: Nanoscale	MEE QCs: Macroscale
Static	Xiao et al. [197] Guo and Li [198] Zhao and Guo [199] Ma et al. [200] Yang and Liu [115] Su et al. [201] Su and Xiao [202] Xin and Xiao [203] Wu et al. [204] Wang et al. [205]	Fan et al. [103] Pi et al. [184] Yang et al. [185] Yu et al. [186] Yang and Li [206] Zhou and Li [187, 188] Zhang et al. [189] Hu et al. [190, 191] Zhang et al. [192] Zhou et al. [193] Li et al. [143, 145]	Not found	Not found
Dynamic	Not found	Yang et al. [194, 195] Ma et al. [152] Panja and Mandal [196]	Not found	Not found

piezoelectricity and elasticity, giving them multifunctional properties not found in conventional crystals. These properties make them attractive materials for sensors, actuators and energy-harvesting applications. Their aperiodic arrangement typically results in brittle fracture, which is further exacerbated by the presence of macro- and nano- scale defects such as cracks, holes and inclusions.These defects act as stress concentrators under coupled mechanical, electric and magnetic loading. A survey of the literature reveals paucity of studies addressing the static and dynamic fracture such materials with defects. While fracture mechanics of classical MEE composites and elastic QCs has advanced in recent years, the complex problem of MEE QCs with macro- and nano- scale defects under coupled loads remains largely unexplored, see Table 6. Bulk MEE QCs exhibiting both intrinsic magnetic ordering and piezoelectric coupling have not been experimentally investigated, though rare-earth icosahedral QCs with long-range magnetic order have been synthesized. Although current manufacturing efforts focus on alloy design, strain / phason engineering, and heterostructures or metamaterials, a fully functional MEE QC has not been reported. From a modeling perspective, strategies developed for PQCs can be adapted to investigate MEE QCs. Continuum mechanics approaches extending classical magneto-electro-elastic constitutive models can account for quasicrystalline symmetries and phason contributions, enabling the analysis of stress, electric, and magnetic fields around defects. Numerical frameworks such as the boundary elements, finite elements, finite

differences, Green's functions and phase-field methods can be systematically applied to study static and dynamic fracture, supporting both their design and potential applications. Overall, while theoretical understanding of PQCs is well developed, MEE QCs remain largely uncharted, both theoretically and experimentally.

6 Conclusions

We conclude this state-of-the-art review with the following observations:

(i) The systematic development of models describing the static as well as the dynamic mechanical behavior of macro- and nano-heterogeneous multifunctional quasicrystals is still ongoing. This effort requires a multidisciplinary approach that spans physics, chemistry, materials science, continuum mechanics and computational methods in order to capture the complexity of these systems. Although current efforts are primarily focused on advancing theoretical and computational models, their reliability and practical relevance ultimately depend on validation procedure using carefully designed experiments, a demanding, costly, and technically challenging endeavor. To characterize both existing QCs and design new ones, an integrated approach is essential that combines experimental findings with numerical simulations based on advanced mechanical models.

(ii) Research on fracture in QCs has so far been dominated by analytical methodologies. While elegant and powerful, these approaches are generally limited to relatively simple configurations, such as infinitely extending homogeneous media without defects, and typically assume idealized conditions such as material linearity. Over the years, a broad range of such methods has been adapted to the QC context, including *potential theory*, *conformal mapping and complex variable techniques*, *generalized Lekhnitskii formalism*, *integral transform methods*, *Fourier-based formulations*, *the Stroh formalism*, and *displacement discontinuity methods*. These techniques have provided valuable insights into the underlying mechanics, but their applicability to realistic geometries, material inhomogeneity and complex defect configurations remains restricted. By contrast, purely numerical methods remain underrepresented. Some progress has been made with *meshless formulations*, which show promise in handling complex crack topologies. However, applications of the *boundary element method (BEM),* despite a well-established track record in fracture analysis of solids, are almost absent within the QC context, largely due to the lack of closed-form fundamental solutions. Similarly, only a few studies use the *finite element method (FEM)*, while *finite difference method (FDM)* applications are limited

to a small number of dynamic fracture problems. Overall, although the analytical tradition has yielded a rich theoretical foundation, the development of robust numerical tools such as those listed above is still at an early stage. Significant opportunities remain for addressing realistic problems involving finite and semi-infinite domains, multiple interacting defects, nonlinear material responses, dynamic loading, different classes of 2D and 3D QCs, and additional complexities such as anisotropy, material gradients, and porosity.

(iii) Most of the fundamental solutions of the governing equations in QC models address static problems of 1D QCs under anti-plane deformation, as summarized in Tables 1–3. Results are scarce for 2D (in-plane) and 3D elastodynamic fundamental solutions, as well as for half-plane and half-space Green's functions in homogeneous or functionally graded QCs, and accounting for anisotropy. The same holds true for available mechanical models (Tables 4, 5), the majority of which address elastic isotropic 1D QCs under anti-plane static loads, typically with a single defect in an otherwise homogeneous infinite domain.

(iv) There are no results for mechanical models of macro- and nano- heterogeneous PQCs and MEE QCs (Table 6). Because of the complex coupling effects between elastic, electric, magnetic, phonon, and phason fields under applied excitations, analytical formulations and numerical solutions of boundary-value problems remain difficult to obtain. While the piezoelectric effect of QCs has been studied to some extent, this research is still far from complete.

Some open questions and challenges for future research are as follows:

(i) *Clarify the physical basis of elastodynamic models*, and specifically existing models include elastic waves based on Bak's theory, elasto-hydrodynamic (diffusive) models based on Lubensky's theory, and wave-telegraph type models. Future work should accumulate more results from mechanical modeling based on these hypotheses and systematically compare theoretical predictions with experimental observations.

(ii) *Develop BEM-based mechanical models,* which depends strongly on deriving families of 2D / 3D, static and dynamic fundamental solutions / Green's functions for elastic, piezoelectric, and magnetopiezoelectric QCs. The BEM offers key advantages such as problem dimensionality reduction, efficient modeling of complex geometries, solution at selective internal points without domain discretization, high accuracy using problem-specific fundamental solutions, natural treatment of infinite boundaries, and precise computation of stress con-

centrations near defects. Leveraging these advantages requires both theoretical developments and efficient numerical implementations.

(iii) *Experimental validation* is necessary since progress in QC modeling must be closely coupled with experimental studies. Validating the predictions of analytical and numerical models through controlled experiments is essential for confirming their accuracy, guiding model refinement and ensuring their practical applicability in real materials and devices.

In sum, the field of mechanical modeling of quasicrystals remains at an early, but exciting stage. Although significant theoretical advances have been made, comprehensive experimental validation and robust numerical frameworks still need to be researched and developed. Addressing the open questions outlined in this work will not only deepen a fundamental understanding of QC mechanics, but also accelerate their application in data sensing, actuation, energy harvesting and other emerging technologies.

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