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FLUID MECHANICS

EFFECT OF TEMPERATURE DEPENDENT VISCOSITY ON THERMAL INSTABILITY OF OLDROYDIAN VISCOELASTIC FLUID LAYER SATURATING POROUS MEDIUM

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ABSTRACT: The effect of temperature-dependent viscosity on the onset of thermal convection in a viscoelastic fluid-saturating porous medium is studied for different cases of combination of rigid and dynamically free boundaries, Darcy-Brinkman-Oldroyd model is considered to investigate the rheological behaviour of the fluid. A necessary condition for the existence of overstability is derived, analytically. The values of critical Darcy-Rayleigh numbers for both stationary and oscillatory convection with linear and exponential viscosity variations (i.e. temperature-dependent viscosity) are computed numerically, using the Galerkin technique. The effects of rheological parameters, modified Darcy number, wave number and variable viscosity parameter on the stability of the system are computed numerically and depicted graphically for each case of combinations of boundary conditions. It is observed that the viscoelastic parameters; stress relaxation time and strain retardation time do not affect the onset of stationary convection, whereas the stress relaxation time has destabilizing effect and strain retardation time has stabilizing effect on the onset of oscillatory convection. It is also found that the temperature-dependent viscosity and modified Darcy number have a stabilizing effect on the onset of stationary as well as oscillatory convection for each case of boundary conditions.

KEY WORDS: Darcy-Brinkman Model, Galerkin technique, Oldroydian Model, Porous medium, Temperature-dependent viscosity, Thermal convection, viscoelastic fluid.

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1 Introduction

Rayleigh-Bnard convection (or Thermal convection) problem with constant viscosity for Newtonian fluid has been extensively studied by many researchers since the experimental and mathematical investigations respectively of the problem by Bnard [1] and Rayleigh [2]. Thermal convection in the domain of hydrodynamic stability theory, is well suited to illustrate many mathematical and physical phenomena and find many engineering and industrial applications. For a detailed view of the Rayleigh-Bnard convection problem, one may refer to Jeffreys [3], Low [4], Pellew and Southwell [5] and Chandrasekhar [6].

From the review of the literature, it is evident that most of the stability investigations of the convective problem have been carried out by taking the viscosity of the fluid as constant, as it imparts an additional mathematical complexity in dealing with the eigenvalue problems in terms of a set of differential equations having variable coefficients. Straughan [7] in the investigation of the stability of a fluid layer with temperature-dependent viscosity reported that the viscosity is more sensitive to the temperature than any other physical properties of the fluid. Further, Torrance & Turcotte [8] worked on the influence of large variations of viscosity on convection in a layer of fluid heated from below and found that with increasing temperature the viscosity of the gases increases, while that of liquids decreases. Some authors, including Trompert and Hansen [9], Stengel et al. [10], Dhiman and Kumar [11] and Dhiman and Sharma [12, 13] have also studied the effect of variable viscosity on the thermal convection problems. From these investigations, we can conclude that the range of the applied temperature in the study of fluid dynamical characteristics, is a crucial parameter; particularly in the fields of study of oceanography, astrophysics, etc.

Jenkins [14] studied the effects of linear and exponential temperature-dependent viscosity variations and observed that the exponential dependence is more realistic for fluids with high viscosity than the linear dependence, for fluids with small values of viscosity. Many authors including; Sekhar and Jayalatha [17] have reported some more general empirical expressions for viscosity variation, however, the linear temperature-dependent viscosity variation and exponential temperature-dependent viscosity variation laws are the only two laws that fit quite well for experimental data on restricted temperature ranges for a wide class of gases and liquids.

Among different kinds of non-Newtonian fluids, viscoelastic fluids are considered to be of considerable importance due to their frequent occurrence in nature. Apart from its rheological importance, many authors have been motivated to investigate the thermal instability problems in the viscoelastic fluid due to its interesting applications. Sokolov and Tanner [16] have investigated the stability problem for a plane layer of a general viscoelastic fluid heated from below and found an oscillating cell structure will be created before the classical (Bnard) steady secondaryflow instability appears. Recently, Sekhar and Jayalatha [17] have studied thermal instability in viscoelastic liquids with temperature-dependent viscosity for different models of shear viscosity as a function of temperature for different combinations of boundary conditions and showed that the existence of oscillatory mode. In a series of papers, Oldroyd [22–24] worked on a set of constitutive equations for viscoelastic fluids to explain some rheological behaviour of some non-Newtonian fluids. From these investigations of the thermal instability problems in viscoelastic fluids, we observed that the principle of exchange of stability, in general, is not valid, and hence the oscillatory motions may appear.

Since Horton and Rogers [18] and Lapwood [19] reported the investigations on thermal convection in a porous medium, many authors in the recent past have studied various dynamical phenomena in a porous medium, due to its relevance in numerous applications; such as geothermal energy utilization, polymer engineering, ceramic processing and enhanced recovery of petroleum reservoirs. The detailed overview of the work devoted the convection in a porous medium is well documented in Ingham and Pop [20], Nield and Bejan [15], Shivakumara et al. [21].

Motivated by the above discussions and taking into account the considerable importance of non-Newtonian fluid saturating porous medium and the significant influence of linear/exponential viscosity variations (i.e. temperature-dependent viscosity variations) on the onset of convection, the purpose of the present work is to investigate the effects of temperature-dependent viscosity on the onset of Rayleigh-Bnard convection in a horizontal porous layer of an Oldroydian viscoelastic fluid model. The different combinations of rigid and dynamically free boundary conditions are considered here. A linear stability analysis based on the normal mode technique is used to derive the eigenvalue problem of this physical configuration. From the general eigenvalue problem, the values of critical Darcy-Rayleigh numbers for both stationary and oscillatory convection with temperature-dependent viscosity, for each case of boundary conditions are computed using the single-term Galerkin technique. The effects of linear and exponential viscosity variations, wave number, stress relaxation parameter, and modified Darcy number on the onset of both stationary as well as oscillatory convection have been computed numerically and depicted graphically for each case of boundary combinations. It is established that the variable viscosity parameter, stress relaxation parameter, strain retardation parameter, and modified Darcy number have significant effects on the onset of convection.

2 Mathematical Model

Consider an infinite horizontal porous layer of an Oldroydian viscoelastic fluid of thickness d in the force field of gravity $\vec{g}(0,0,-g)$. The upper plane at $z=d$ and the lower plane at $z=0$ are maintained at constant temperatures T_1 and T_0 $(T_0 > T_1)$ respectively in a porous medium, thus maintaining a uniform adverse temperature gradient (β) tending to increase the density of the fluid in vertical direction z. The fluid is assumed to be incompressible (Boussinesq) and the viscosity is assumed to depend on the temperature. The Darcy-Brinkman-Oldroyd model, which includes the time derivative, is employed as a momentum equation. The governing equations

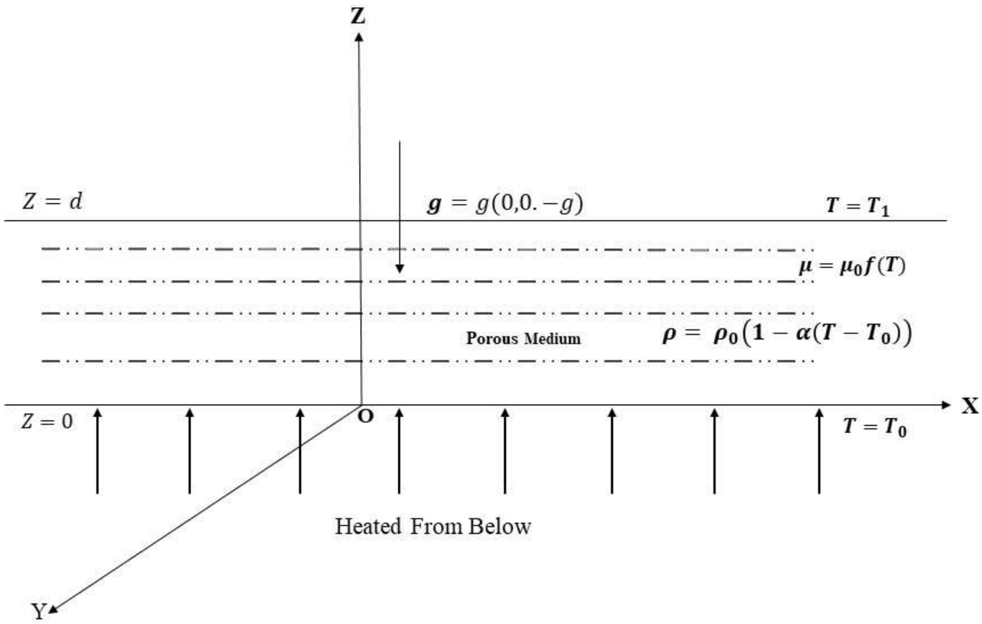


Fig. 1: Physical Configuration

for the convection inside the porous layer have the following form;

(1) $$\nabla\vec{q} = 0,$$

(2) $$\Big(1+\tau_1\frac{\partial}{\partial t}\Big)\Big(\frac{\rho_0}{\epsilon}\frac{D\vec{q}}{Dt}+\nabla P+\rho g\hat{k}\Big) = \Big(1+\tau_2\frac{\partial}{\partial t}\Big)\frac{\partial}{\partial x_j}\Big[\frac{\mu_0 f}{\epsilon}\Big(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\Big)\Big],$$

(3) $$\rho_0 C_v\frac{DT}{Dt} = k_1\nabla^2 T,$$

(4) $$\rho = \rho_0\left[1-\alpha\left(T-T_0\right)\right].$$

In the above equations, $\vec{q}\,[=(u,v,w)]$ is the velocity vector, T is the temperature, P is the pressure, $\hat{k}$ is the unit vector in the z-direction, $\frac{D}{Dt} \equiv \left(\frac{\partial}{\partial t} + \vec{q}\cdot\nabla\right)$ is the material derivative, τ_1 and τ_2, respectively, are stress relaxation time and strain retardation time parameters, ρ and ρ_0 are the densities at the temperatures T and T_0, respectively, α is the coefficient of thermal expansion, g is the acceleration due to gravity, ϵ is porosity, $\mu(=\mu_0 f(T))$ is the temperature-dependent viscosity, k_1 is the thermal conductivity, C_v is the specific heat at constant volume. To investigate the instability of the equilibrium system, let us add small perturbations to the initial state solution and the perturbed quantities are denoted as

$$\begin{gathered}\vec{q} = \vec{q_b}(0,0,0) + \vec{V}, \quad P = P_b(z) + \delta p,\\ T = T_b(z) + \theta, \quad \rho = \rho_0\left[1 - \alpha\{T_b(z) + \theta - T_0\}\right],\end{gathered} \tag{5}$$

where $\vec{V}\,(u,v,w)$, δp and θ are the respective perturbations in the initial velocity components, pressure P and temperature T and $\vec{q_b}$, $P_b(z)$ and $T_b(z)$ are the basic state solutions. Now following the usual steps of linear stability, substituting perturbed quantities in equations (1)-(4), we obtain the following linearised perturbation equations:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{6}$$

$$\begin{aligned}\left(1+\tau_1\frac{\partial}{\partial t}\right)\left(\frac{\rho_0}{\epsilon}\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\delta p\right) &= \\ &= \frac{\mu_0}{\epsilon}\left(1+\tau_2\frac{\partial}{\partial t}\right)\left[f\left(\nabla^2 - \frac{\epsilon}{K}\right)u + \frac{\partial f}{\partial z}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right],\end{aligned} \tag{7}$$

$$\begin{aligned}\left(1+\tau_1\frac{\partial}{\partial t}\right)\left(\frac{\rho_0}{\epsilon}\frac{\partial v}{\partial t} + \frac{\partial}{\partial y}\delta p\right) &= \\ &= \frac{\mu_0}{\epsilon}\left(1+\tau_2\frac{\partial}{\partial t}\right)\left[f\left(\nabla^2 - \frac{\epsilon}{K}\right)v + \frac{\partial f}{\partial z}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right],\end{aligned} \tag{8}$$

$$\begin{aligned}\left(1+\tau_1\frac{\partial}{\partial t}\right)\left(\frac{\rho_0}{\epsilon}\frac{\partial w}{\partial t} + \frac{\partial}{\partial z}\delta p - g\alpha\theta\rho_0\right) &= \\ &= \frac{\mu_0}{\epsilon}\left(1+\tau_2\frac{\partial}{\partial t}\right)\left[f\left(\nabla^2 - \frac{\epsilon}{K}\right)w + 2\frac{\partial f}{\partial z}\frac{\partial w}{\partial z}\right],\end{aligned} \tag{9}$$

$$\frac{\partial \theta}{\partial t} = \beta w + \kappa\nabla^2\theta. \tag{10}$$

Here, K is the permeability of the porous medium and $\kappa = \frac{k_1}{\rho_0 C_v}$ is the effective thermal diffusivity.

The equations(6)-(10) must seek solutions subject to certain boundary conditions based upon the nature of the bounding surfaces, which shall be either rigid (with the no-slip condition) or dynamically free (where the tangential stresses vanish). Hence, the relevant boundary conditions are

$$(11)\qquad \begin{aligned} w = 0 = \theta = \frac{\partial^2 w}{\partial z^2} \quad \text{at} \quad z = 0 \quad (\text{or} \ \ z = d) \\ w = 0 = \theta = \frac{\partial w}{\partial z} \quad \text{at} \quad z = 0 \quad (\text{or} \ \ z = d)\,, \end{aligned}$$

on dynamically free and rigid surfaces respectively. Now, eliminating δp amongst the equations(7)-(9), we have

$$(12)\qquad \Big(1 + \tau_1 \frac{\partial}{\partial t}\Big)\Big[\frac{\rho_0}{\epsilon}\frac{\partial}{\partial t}\nabla^2 w - \rho_0 g \alpha \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\theta\Big] = \mu_0 \Big(1 + \tau_2 \frac{\partial}{\partial t}\Big)$$
$$\times \Big[\Big(\frac{1}{\epsilon}\nabla^2 - \frac{1}{K}\Big) f \nabla^2 w - \frac{1}{\epsilon}\Big(\frac{d^2 f}{dz^2}\Big(\nabla^2 - 2\frac{\partial^2}{\partial z^2}\Big) w + 2\frac{df}{dz}\frac{\partial}{\partial z}\left(\nabla^2 w\right)\Big)\Big]$$
$$- \mu_0 \Big(1 + \tau_2 \frac{\partial}{\partial t}\Big)\Big[\frac{1}{K}\frac{df}{dz}\frac{dw}{dz}\Big].$$

Now, using the *normal mode analysis* in equations (12) and (10) by assuming the time dependence periodic solution in the horizontal plane of the form;

$$(13)\qquad [u, v, w, \theta] = [u(z), v(z), w(z), \theta(z)] \exp\left[i\left(k_x x + k_y y\right) + nt\right]$$

and using the following non-dimensional quantities in the resulting equations:

$$\hat{z} = \frac{z}{d}, \quad \hat{t} = \frac{\kappa t}{d^2}, \quad \hat{w} = \frac{dw}{\nu}, \quad \hat{a} = kd, \quad \hat{D} = d\frac{d}{dz}, \quad \hat{\sigma} = \frac{nd^2}{\kappa}, \quad \hat{\lambda_1} = \frac{\tau_1 \kappa}{d^2},$$
$$Da = \frac{K}{\epsilon d^2}, \quad \hat{\lambda_2} = \frac{\tau_2 \kappa}{d^2}, \quad \hat{\theta} = \frac{\kappa \theta}{\beta \nu d}, \quad \Pr_D = \frac{\nu}{\kappa Da}, \quad \nu = \frac{\mu_0}{\rho_0}$$

we have obtained the following system of non-dimensional linearized perturbation equations, respectively:

$$(14)\qquad \frac{1+\lambda_2\sigma}{1+\lambda_1\sigma}\Big[Da\left\{f\left(D^2 - a^2\right)^2 + D^2 f\left(D^2 + a^2\right) + 2DfD\left(D^2 - a^2\right)\right\}$$
$$- \left\{f\left(D^2 - a^2\right) + DfD\right\}\Big] w - \frac{\sigma}{\Pr_D}\left(D^2 - a^2\right) w = R_D a^2 \theta$$

$$(15)\qquad \left(D^2 - a^2 - \sigma\right)\theta = -w.$$

In the above equations, n is the growth rate, k_x and k_y are the wave numbers along x and y directions, respectively, Da is the modified Darcy number, λ_1 is non-dimensional stress relaxation time, λ_2 is the non-dimensional strain retardation time, $\mathrm{Pr}_D (= \mathrm{Pr}/Da)$ is the Darcy-Prandtl number, Pr is thermal prandtl number, a is the non-dimensional wave number, σ is the non-dimensional *complex growth rate* and $R_D (= \dfrac{g\alpha\beta d^4 Da}{\kappa\nu} = R \cdot Da)$ is Darcy-Rayleigh number, R is the thermal Rayleigh number.

The non-dimensional boundary conditions (11) in view of (13), can be written as

Case 1: *Both boundaries dynamically free*

$$w = 0 = \theta = D^2 w \quad \text{at} \quad z = 0 \ \text{ and } \ z = 1 \tag{16}$$

Case 2: *Both boundaries rigid*

$$w = 0 = \theta = Dw \quad \text{at} \quad z = 0 \ \text{ and } \ z = 1 \tag{17}$$

Case 3: *One rigid and other dynamically free boundary*

$$\begin{aligned} w = 0 = \theta = Dw \quad &\text{at} \quad z = 0 \ \ (\text{or } \ z = 1) \\ w = 0 = \theta = D^2 w \quad &\text{at} \quad z = 1 \ \ (\text{or } \ z = 0) \end{aligned} \tag{18}$$

Further, the caps have been dropped from the above equations for convenience in writing. The system of equations (14)-(15) together with boundary conditions (16)-(18) constitutes an eigen value problem for the growth rate σ. The growth rate σ is in general a complex quantity such that $\sigma = \sigma_r + i\sigma_i$. The system with $\sigma_r < 0$ is always stable, while for $\sigma_r > 0$ it will become unstable. If $\sigma_r = 0 \Rightarrow \sigma_i = 0$, the principle of exchange of stability is valid and if $\sigma_r = 0 \nRightarrow \sigma_i = 0$, then we have the case of overstability. Further, if σ is real, then $\sigma = 0$ will separate the stable and unstable modes and there will be the exchange of stabilities.

Remark: It is to be noted that the above eigenvalue problem described by equations (14)-(15) together with boundary conditions (16)-(18) yield the eigenvalue problem governing Rayleigh-B$\acute{e}$nard convection in viscoelastic fluid saturating porous medium Siddheshwar and Krishna [25], if we take $\mu = \mu_0$ for $f(T) = 1$ (constant viscosity). Also, the eigenvalue problems governing Rayleigh-B$\acute{e}$nard convection in viscoelastic fluid without porous medium and Rayleigh-B$\acute{e}$nard convection for ordinary Newtonian fluid can be easily deduced by ignoring the respective terms in equations (14)-(15).

3 Stability Analysis

3.1 Principle of exchange of stabilities

First of all, we shall investigate whether the principle of exchange of stabilities (PES) is valid for this general problem, or not. We shall follow the *Pellew and Southwell method* [5] to prove this.

Multiplying both sides of the equation (14) by w^* (the complex conjugate of w) and using w^* obtained from the complex conjugate of the equation (15), the resulting equation is integrating by parts a suitable number of times using relevant boundary conditions (16)-(18) over the vertical range of z, we have

$$
\begin{aligned}
(19)\quad & Da\left[\int_0^1 f\left[|D^2w|^2+2a^2|Dw|^2+a^4|w|^2\right]dz+a^2\int_0^1\left(D^2f\right)|w|^2dz\right]\\
&+\int_0^1 f(|Dw|^2+a^2|w|^2)dz+\left(\frac{1+\lambda_1\sigma}{1+\lambda_2\sigma}\right)\frac{\sigma}{\mathrm{Pr}_D}\int_0^1(|Dw|^2+a^2|w|^2)dz\\
&=R_Da^2\left(\frac{1+\lambda_1\sigma}{1+\lambda_2\sigma}\right)\int_0^1(|D\theta|^2+a^2|\theta|^2+\sigma^*|\theta|^2)dz.
\end{aligned}
$$

The above equation after some simplifications yields the following equation:

$$
\begin{aligned}
(20)\quad & Da\left[\int_0^1 f[|D^2w|^2+2a^2|Dw|^2+a^4|w|^2]dz+a^2\int_0^1(D^2f)|w|^2dz\right]\\
&+\int_0^1 f(|Dw|^2+a^2|w|^2)dz\\
&+\left\{\frac{1+\lambda_1\lambda_2|\sigma|^2+\lambda_1\sigma+\lambda_2\sigma^*}{1+\lambda_2\left(\sigma+\sigma^*\right)+\lambda_2^2|\sigma|^2}\right\}\left(\frac{\sigma}{\mathrm{Pr}_D}\right)\int_0^1(|Dw|^2+a^2|w|^2)dz\\
&=R_Da^2\left\{\frac{1+\lambda_1\lambda_2|\sigma|^2+\lambda_1\sigma+\lambda_2\sigma^*}{1+\lambda_2(\sigma+\sigma^*)+\lambda_2^2|\sigma|^2}\right\}\int_0^1(|D\theta|^2+a^2|\theta|^2+\sigma^*|\theta|^2)dz.
\end{aligned}
$$

Now, substituting, $\sigma=\sigma_r+i\sigma_i$ in above equation (20), which after some algebraic manipulation, yields

(21)
$$
\begin{aligned}
&Da\Big[\int_0^1 f[|D^2w|^2+2a^2|Dw|^2+a^4|w|^2]dz + a^2\int_0^1 (D^2f)|w|^2dz\Big]\\
&+\int_0^1 f(|Dw|^2+a^2|w|^2)dz+\int_0^1(|Dw|^2+a^2|w|^2)dz\\
&\times\frac{[\sigma_r(1+\lambda_1\lambda_2|\sigma|^2)+(\sigma_r^2-\sigma_i^2)\lambda_1+\lambda_2|\sigma|^2+i\sigma_i(1+\lambda_1\lambda_2|\sigma|^2+2\sigma_r\lambda_1)]}{\mathrm{Pr}_D(1+2\sigma_r\lambda_2+\lambda_2^2|\sigma|^2)}\\
&=\frac{R_Da^2}{1+2\sigma_r\lambda_2+\lambda_2^2|\sigma|^2}\{[(1+\lambda_1\lambda_2|\sigma|^2)\int_0^1(|D\theta|^2+a^2|\theta|^2)dz\\
&+\lambda_1|\sigma|^2\int_0^1|\theta|^2dz+\sigma_r(\lambda_1+\lambda_2)\int_0^1(|D\theta|^2+a^2|\theta|^2)dz\\
&+\sigma_r(1+\lambda_1\lambda_2|\sigma|^2)\int_0^1|\theta|^2dz+(\sigma_r^2-\sigma_i^2)\lambda_2\int_0^1|\theta|^2dz]\\
&+i\sigma_i[(\lambda_1-\lambda_2)\int_0^1(|D\theta|^2+a^2|\theta|^2)dz-(1+\lambda_1\lambda_2|\sigma|^2+2\sigma_r\lambda_2)\int_0^1|\theta|^2dz]\}.
\end{aligned}
$$

Comparing the imaginary parts on both sides of equation (21) and cancelling $\sigma_i \neq 0$ throughout, we have

(22)
$$
\begin{aligned}
&\frac{(1+\lambda_1\lambda_2|\sigma|^2+2\sigma_r\lambda_1)}{\mathrm{Pr}_D}\int_0^1\left(|Dw|^2+a^2|w|^2\right)dz\\
&=R_Da^2\left[(\lambda_1-\lambda_2)\int_0^1(|D\theta|^2+a^2|\theta|^2)dz-(1+\lambda_1\lambda_2|\sigma|^2+2\sigma_r\lambda_2)\int_0^1|\theta|^2dz\right].
\end{aligned}
$$

Now, multiplying equation (15) by θ^*, integrating it over the vertical range of z a suitable number of times using boundary conditions (16)-(18), and the real parts of the resulting equation yields the following inequality

(23)
$$
\begin{aligned}
&\int_0^1\left(|D\theta|^2+a^2|\theta|^2+\sigma_r|\theta|^2\right)dz\\
&\qquad=\text{real part of }\left\{\int_0^1\theta^* w dz\right\}\le\left|\int_0^1\theta^* w dz\right|\le\int_0^1|\theta||w|dz.
\end{aligned}
$$

Let us assume that, $\sigma_r \geq 0$, the above inequality after using *Schwartz's inequality* yields

$$\int_0^1 \left(|D\theta|^2 + a^2|\theta|^2\right) dz < \left\{ \int_0^1 |\theta|^2 dz \right\}^{\frac{1}{2}} \left\{ \int_0^1 |w|^2 dz \right\}^{\frac{1}{2}}. \tag{24}$$

From the above inequality, we can have

$$\left\{ \int_0^1 |\theta|^2 dz \right\}^{\frac{1}{2}} < \frac{1}{a^2} \left\{ \int_0^1 |w|^2 dz \right\}^{\frac{1}{2}}. \tag{25}$$

Using inequality (25) in inequality (24), we have

$$\int_0^1 \left(|D\theta|^2 + a^2|\theta|^2\right) dz < \frac{1}{a^2} \int_0^1 |w|^2 dz. \tag{26}$$

Now, using inequality (26) in inequality (25), we have

$$\frac{\left(1 + \lambda_1\lambda_2|\sigma|^2 + 2\sigma_r\lambda_1\right)}{\Pr_D} \int_0^1 \left(|Dw|^2 + a^2|w|^2\right) dz + R_D a^2 [\left(1 + \lambda_1\lambda_2|\sigma|^2 + 2\sigma_r\lambda_2\right) \int_0^1 |\theta|^2 dz] < R_D \left(\lambda_1 - \lambda_2\right) \int_0^1 |w|^2 dz. \tag{27}$$

Since, w vanishes at $z = 0$ and $z = 1$, therefore using *Rayleigh Ritz inequality*, we have

$$\int_0^1 |Dw|^2 dz \geq \pi^2 \int_0^1 |w|^2 dz. \tag{28}$$

Using inequality (28) in inequality (27), which after some rearrangement takes the following form

$$\left[\frac{1}{\Pr_D} - \frac{\lambda_1 R_D \left(1 - Q\right)}{\pi^2}\right] \int_0^1 |Dw|^2 dz + \frac{(1+\lambda_1\lambda_2|\sigma|^2+2\sigma_r\lambda_1)}{\Pr_D} \int_0^1 |Dw|^2 dz + \frac{(1+\lambda_1\lambda_2|\sigma|^2+2\sigma_r\lambda_1)a^2}{\Pr_D} \int_0^1 |w|^2 dz + R_D a^2 \left[\left(1 + \lambda_1\lambda_2|\sigma|^2 + 2\sigma_r\lambda_2\right) \int_0^1 |\theta|^2 dz\right] < 0. \tag{29}$$

Since, $\sigma_r \geq 0$, the above inequality clearly implies that for $0 < Q \left(= \frac{\lambda_2}{\lambda_1}\right) < 1$ and $\lambda_1 > 0$,

$$R_D > \frac{\pi^2}{\Pr_D \lambda_1 \left(1 - Q\right)}. \tag{30}$$

The condition (30) (true for $\sigma_r \geq 0$ and $\sigma_i \neq 0$) represents a necessary condition for overstability and hence, if $\sigma_r \geq 0$ and $R_D \leq \dfrac{\pi^2}{\mathrm{Pr}_D\, \lambda_1 (1-Q)}$, then we must have $\sigma_i = 0$, which is a sufficient condition for the validity of principle of exchange of stabilities (*PES*).

3.2 Expressions for Rayleigh numbers

We shall use a single-term Galerkin method to find the value of the Darcy-Rayleigh number for each case of boundary conditions (16)-(18). Let us take a single-term in the expansion of w and θ as

$$w = Aw_1(z), \qquad \theta = B\theta_1(z), \tag{31}$$

where, w_1 and θ_1 are suitably chosen trial functions which satisfy the respective boundary conditions (16)-(18), and A and B are constants.

Substituting the values of w and θ from equation (31) in equations (14)-(15), multiplying the resulting equations by w_1 and θ_1 respectively, integrating by parts a suitable number of times and using the relevant boundary conditions (16)-(18), we obtain the following relation between Darcy-Rayleigh number R_D and wave number a (the cell size):

$$R_D = \frac{X_4 + \sigma X_5}{a^2 X_3^2}\left[(DaX_1 + X_6)\left(\frac{1+\lambda_2\sigma}{1+\lambda_1\sigma}\right) + \frac{\sigma}{\mathrm{Pr}_D}X_2\right], \tag{32}$$

where

$$X_1 = \int_0^1 f\left[\left(D^2 w_1\right)^2 + 2a^2\left(Dw_1\right)^2 + a^4 w_1^2\right]dz + a^2\int_0^1 w_1^2 D^2 f dz,$$

$$X_2 = \int_0^1 \left[\left(Dw_1\right)^2 + a^2 w_1^2\right]dz, \quad X_3 = \int_0^1 \theta_1 w_1 dz, \tag{33}$$

$$X_4 = \int_0^1 [(D\theta_1)^2 + a^2\theta_1^2]dz, \quad X_5 = \int_0^1 \theta_1^2 dz, \quad X_6 = \int_0^1 f[(Dw_1)^2 + a^2 w_1^2]dz.$$

It is to be mentioned here that the above expression (32) for Darcy-Rayleigh number is general and is valid for all cases of boundary conditions.

Case of stationary convection

In the above analysis, it is established that the principle of exchange of stabilities may be valid, which yields that the onset of convection in this general problem shall be through stationary mode. Hence, setting $\sigma = 0$ in expression (32), we have

$$R_D^{\mathrm{st}} = \frac{(DaX_1 + X_6)\,X_4}{a^2 X_3^2}, \tag{34}$$

where, X_1, X_3, X_4 and X_6 are already defined in (33).

It is remarkable to note that the expression (34) is valid for all cases of boundary conditions and arbitrary functions of viscosity variation $f(z)$.

We shall, now, obtain the values of Darcy-Rayleigh numbers and consequently, the values of the critical Darcy-Rayleigh number for the stationary case from expression (34), for different cases of boundary conditions, using the following *linear* and *exponential* non-dimensional viscosity variation law (cf. [11]):

$$f = (1 + \delta z) \qquad \text{and} \qquad f = e^{\delta z}, \tag{35}$$

where, $\delta = \gamma\beta d$ is the viscosity variation parameter and γ is the coefficient of viscosity variation.

As mentioned (see also [11]), the viscosity variations laws are functions of magnitude of β (the temperature gradient across the layer), which means that for a large temperature gradient across the layer, the viscosity shall be smaller.

We shall consider the following trial functions (Table 1), appropriate for various cases of boundary conditions (*cf.* Nield [15]).

Table 1

Boundary combinations	Velocity(w)	Temperature (θ)
Case 1: free-free	$z^4 - 2z^3 + z$	$z\,(z-1)$
Case 2: rigid-free	$2z^4 - 5z^3 + 3z^2$	$z\,(z-1)$
Case 3: rigid-rigid	$z^4 - 2z^3 + z^2$	$z\,(z-1)$

It is to be mentioned here that for other combinations of boundary conditions; Case 4: (18); *i.e. upper rigid and lower free boundaries* is not treated separately in this analysis, as it yields almost the same numerical values for Rayleigh numbers and hence are omitted here for the sake of compactness.

Case1: When both boundaries are dynamically free. Using the appropriate trial functions for the present case from Table 1 and *exponential* viscosity variation law (35), we obtain the following Darcy-Rayleigh number $R_D^{\rm ff}$ for *exponential* viscosity variation:

$$\begin{aligned} R_D^{\rm ff} = \frac{5880}{289a^2\delta^9}(10+a^2)\Big[2a^4 Da\Big(&-20160 - 10080\delta - 1440\delta^2 + 120\delta^3 \\ &+ 48\delta^4 - \delta^6 + e^{\delta}(20160 - 10080\delta + 1440\delta^2 + 120\delta^3 - 48\delta^4 + \delta^6)\Big) \\ &+ \delta^2\Big[-11520 - 5760\delta - 864(1+4Da)\delta^2 - 48(-1+36Da)\delta^3 \\ &+ (24 - 288Da)\delta^4 - \delta^6 + e^{\delta}\Big(11520 - 5760\delta + 864(1+4Da)\delta^2 \\ &- 48(-1+36Da)\delta^3 + 24(-1+12Da)\delta^4 + \delta^6\Big)\Big] + \end{aligned} \tag{36}$$

$$
\begin{aligned}
&+2a^2\Big[-20160-10080\delta-1440(1+22Da)\delta^2-120(-1+132Da)\delta^3\\
&\quad-48(-1+48Da)\delta^4+168Da\delta^5+(-1+72Da)\delta^6-2Da\delta^8\\
&\quad+e^\delta\Big(20160-10080\delta+1440(1+22Da)\delta^2-120(-1+132Da)\delta^3\\
&\qquad+48(-1+48Da)\delta^4+168Da\delta^5+(1-72Da)\delta^6+2Da\delta^8\Big)\Big]\Bigg].
\end{aligned}
$$

The values of Darcy-Rayleigh number ($R_D^{\rm ff}$) for the case of *linear* variation of viscosity can be obtained from expression(36) as a first order approximation for small viscosity variations and is given by

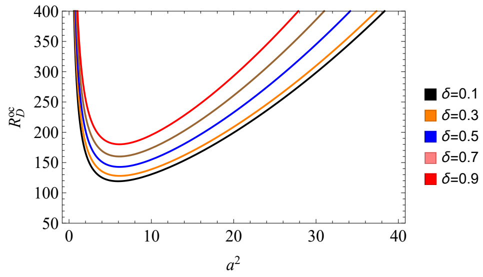
$$
\begin{aligned}
(37)\quad R_D^{\rm ff}=\frac{14}{867a^2}(10+a^2)\Big[&612-54\delta+a^2(62+17\delta)\\
&+Da\Big(-3024(-2+\delta)+a^4(62+17\delta)-72a^2(-17+19\delta)\Big)\Big].
\end{aligned}
$$

Now, for *linear* viscosity variation, the critical value of the wave number (*the critical cell size*) is determined from the condition, $\dfrac{d(R_D^{\rm ff})}{da^2}=0$, which yields a 3^{rd} degree algebraic equation in a^2 and it is cumbersome to solve it analytically for any positive root. Therefore, we have computed the roots for different values of δ using *Mathematica* software, which gives us the values of critical wave number a_c. Hence, using these values of critical wave numbers, the values of critical Darcy-Rayleigh numbers for *linear* viscosity variation are obtained and are presented in Table 2. For the case of exponential viscosity, the variation of $R_D^{\rm st}$ *vs.* a^2 is shown graphically in Fig. 2, for different values of viscosity variation δ.

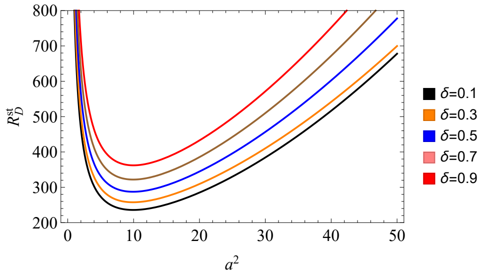
Case 2: When lower boundary rigid and upper boundary free. Using the suitable trial functions for the present case of boundary conditions from Table 1 and

Table 2: Values of critical wave number a_c^2 and Darcy-Rayleigh number $R_D^{\rm st}$ for different values of δ and fixed values of $Da=0.1, \lambda_1=0.7, \lambda_2=0.1, {\rm Pr}_D=10$.

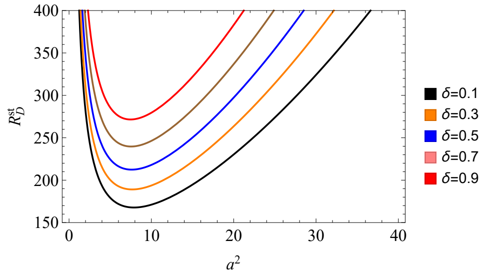
Linear Viscosity Variation for Stationary Convection						
δ	a^2	$R_D^{\rm ff}$	a^2	R_D^{rr}	a^2	R_D^{rf}
0	6.14192	109.629	9.89106	220.672	7.77571	160.103
0.1	6.14192	115.111	9.89106	231.706	7.72252	167.79
0.3	6.14192	126.074	9.89106	253.773	7.63043	183.151
0.5	6.14192	137.037	9.89106	275.841	7.55344	198.496
0.7	6.14192	148	9.89106	297.908	7.48813	213.83
0.9	6.14192	158.963	9.89106	319.975	7.432	229.154



(a) free-free



(b) rigid-rigid



(c) rigid-free

Fig. 2: Plots of R_D^{st} versus a^2 for *exponential* viscosity variations with different values of δ Parameter values: $Da = 0.1$, $\text{Pr}_D = 10$.

proceeding analogously as in Case 1 above, we have obtained the values of critical wave numbers a_c and critical Darcy-Rayleigh numbers for particular values of δ, are presented in Table 2 and are shown graphically in Fig. 2.

Case 3: When both boundaries are rigid. Following the analogous steps followed in Case 1 and Case 2 of boundary conditions above, the values obtained are presented in Table 2 and are shown graphically in Fig. 2.

CASE OF OSCILLATORY CONVECTION

As discussed earlier, for the oscillatory mode instability; the imaginary part of σ, i.e. $\sigma_i \neq 0$. Hence, putting $\sigma = i\omega$ in expression (32), we get

$$R_D = \frac{1}{a^2 X_3^2}\left[(DaX_1 + X_6)\left(N_1 X_4 + \omega^2 N_2 X_5\right) - \frac{\omega^2 X_2 X_5}{\text{Pr}_D}\right] + \frac{i\omega}{a^2 X_3^2} N_3 \tag{38}$$

where,

$$N_1 = \frac{1 + \omega^2 \lambda_1^2 Q}{1 + \omega^2 \lambda_1^2}, \quad N_2 = \frac{\lambda_1 (1 - Q)}{1 + \omega^2 \lambda_1^2}, \quad Q = \frac{\lambda_2}{\lambda_1}, \tag{39}$$

$$N_3 = (DaX_1 + X_6)(N_1 X_5 - N_2 X_4) + \frac{X_2 X_4}{\text{Pr}_D}. \tag{40}$$

The above expression (38) implies that for real R_D; $N_3 = 0$, for given $\omega \neq 0$ (oscillatory case). This condition yields the following expressions for the frequency of oscillation and oscillatory Darcy-Rayleigh number:

$$\omega^2 = -\frac{X_2X_4 + \mathrm{Pr}_D\,(DaX_1 + X_6)\,[X_5 + \lambda_1 X_4\,(Q-1)]}{\lambda_1^2\,[\mathrm{Pr}_D\,Q\,(DaX_1 + X_6)\,X_5 + X_2X_4]}, \tag{41}$$

$$R_D^{\mathrm{oc}} = \frac{1}{a^2X_3^2}\left[(DaX_1 + X_6)\left(N_1X_4 + \omega^2 X_5 N_2\right) - \frac{\omega^2 X_2 X_5}{\mathrm{Pr}_D}\right]. \tag{42}$$

The expressions (41) and (42) are the same as obtained by Siddheshwar and Krishna [25] for the thermal convection problem in a viscoelastic fluid-filled high-porosity medium with non-uniform basic temperature gradient, for constant viscosity (with $\delta = 0$).

Further, it is clear from expression (41) that for oscillation to occur, the value of w^2 must be positive i.e. $Q < 1$ ($\lambda_2 < \lambda_1$) otherwise, PES shall be valid. this is the same condition as derived analytically in Section 3.1.

In the following analysis, we shall now compute the value of Darcy-Rayleigh number, at which the onset of oscillatory instability sets from expression (42), for each case of boundary conditions for both *linear* and *exponential* viscosity variations.

Case 1: Both boundaries dynamically free. Using the appropriate trial functions for the present case from Table 1 and *exponential* viscosity variation law (35), we obtain the following numerical values of integrals X_i (i= 1 to 6):

$$\begin{aligned} X_1 = \frac{2}{\delta^9}\Big\{ & 144\delta^4[(\delta^2 - 6\delta + 12)e^{\delta} - (\delta^2 + 6\delta + 12) \\ & + 2a^2\delta^2\Big[(\delta^6 - 36\delta^4 + 84\delta^3 - 7920\delta + 15840)e^{\delta} \\ & - (15840 + 7920\delta + 1152\delta^2 - 84\delta^3 - 36\delta^4 + \delta^6)\Big] \\ & + a^4\Big[(\delta^6 - 48\delta^4 + 120\delta^3 + 1440\delta^2 - 10080\delta + 20160)e^{\delta} \\ & + (-20160 - 10080\delta - 1440\delta^2 + 120\delta^3 + 48\delta^4 - \delta^6)\Big]\Big\}, \end{aligned} \tag{43}$$

$$X_2 = \frac{31a^2 + 306}{630}, \quad X_3 = \frac{-17}{420}, \quad X_4 = \frac{10 + a^2}{30}, \quad X_5 = \frac{1}{30}, \tag{44}$$

$$\begin{aligned} X_6 = \frac{1}{\delta^9}\Big\{ & \delta^2[\left(\delta^6 - 24\delta^4 + 48\delta^3 + 864\delta^2 - 5760\delta + 11520\right)e^{\delta} \\ & - \left(\delta^6 - 24\delta^4 - 48\delta^3 + 864\delta^2 + 5760\delta + 11520\right)] \\ & + 2a^2[\left(\delta^6 - 48\delta^4 + 120\delta^3 + 1440\delta^2 - 10080\delta + 20160\right)e^{\delta} \\ & - \left(\delta^6 - 48\delta^4 - 120\delta^3 + 1440\delta^2 + 10080\delta + 20160\right)]\Big\}. \end{aligned} \tag{45}$$

The values of definite integrals X_i (i = 1 to 6) for the case of *linear* variation of viscosity can be reduced from *exponential* viscosity variation as a first-order approximation for small viscosity variations and are given by

$$X_1 = \frac{a^4(17\delta+62) - 72a^2(19\delta-17) - 3024(\delta-2)}{1260},$$
$$X_6 = \frac{a^2(17\delta+62) - 54\delta + 612}{1260} \tag{46}$$

and rest of the integrals X_i (i = 2 to 5) will remain unchanged. Using the above obtained values of definite integrals X_i (i = 1 to 6) in the expressions (41) and (42), we can have the expressions for Darcy-Rayleigh number R_D^{oc} for *exponential* and *linear* variation of viscosity. Now, computing the minimum numerical values of these expressions for different values of δ and finding the minimum of each of the obtained values of R_D^{oc} using the calculated value of the critical value of wave number. During this process, we get a 12^{th} degree algebraic equation in a^2, which is cumbersome to handle analytically for getting a positive root. Hence, we have solved it using *Mathematica* software and the values of critical wave number a_c, and consequently the values of critical Darcy-Rayleigh number for *linear* viscosity variation are obtained for a particular set of values of δ and are presented in Table 3. For the case of exponential viscosity, the variation of R_D^{oc} *vs.* a^2 is shown graphically in Fig. 3, for different values of viscosity variation δ.

Case 2: When lower boundary rigid and upper boundary free. Using the suitable trial functions for the present case of boundary conditions from Table 1 and proceeding analogously as in Case 1 above, we have obtained the values of critical wave numbers a_c and critical Darcy-Rayleigh numbers for particular values of δ, are presented in Table 3 and are shown graphically in Fig. 3.

Table 3: Values of critical wave number a_c^2 and Darcy-Rayleigh number R_D^{oc} for different values of δ and fixed values of $Da = 0.1, Q = 1/7, \lambda_1 = 0.7, \lambda_2 = 0.1, \text{Pr}_D = 10$.

	Linear Viscosity Variation for Oscillatory Convection					
δ	a^2	R_D^{ff}	a^2	R_D^{rr}	a^2	R_D^{rf}
0	7.633	25.372	11.689	44.562	9.369	34.501
0.1	7.620	26.384	11.713	46.556	9.316	35.932
0.3	7.595	28.418	11.759	50.565	9.224	38.808
0.5	7.577	30.469	11.780	52.581	9.184	40.253
0.7	7.563	32.534	11.841	58.669	9.083	44.618
0.9	7.551	34.614	11.878	62.759	9.027	47.551

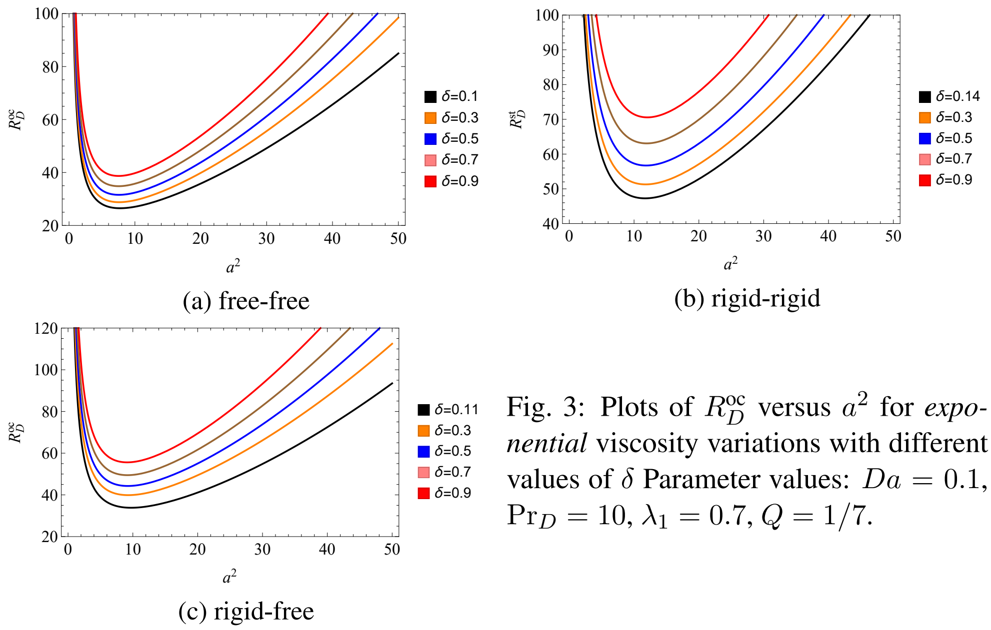


Fig. 3: Plots of R_D^{oc} versus a^2 for *exponential* viscosity variations with different values of δ Parameter values: $Da = 0.1$, $\text{Pr}_D = 10$, $\lambda_1 = 0.7$, $Q = 1/7$.

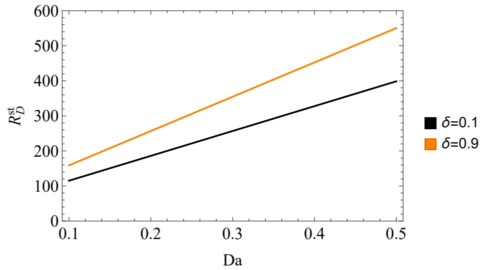
Case 3: When both boundaries are rigid. Following the analogous steps followed in Case 1 and Case 2 of boundary conditions above, the values obtained are presented in Table 3 and are shown graphically in Fig. 3.

4 Results and Discussion

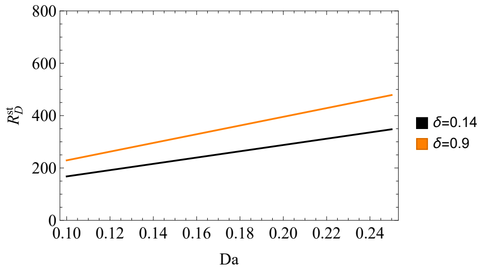
In Table 2, the values of critical wave number a_c^2 and critical Darcy-Rayleigh number for stationary convection with *linear* viscosity variation and the fixed values of other parameters are presented for all cases of boundary conditions.

It is clear from the values presented in Table 2 that the critical wave number (the cell size) is independent of temperature-dependent viscosity δ. From Tables 2-3, it is observed that the values of the critical Darcy-Rayleigh number increase for increasing values of δ for all cases of boundary conditions, which yields that the temperature-dependent viscosity has stabilizing effect on the onset of both stationary and oscillatory convection. From the analysis of the values computed for Darcy-Rayleigh numbers for *exponential* viscosity variation for both stationary and oscillatory convection, we found that it follows analogous trends and variation concerning the parameters as shown in the case of *linear* viscosity variation in Tables 2-3, hence the tables of values for the *exponential* case are omitted here for the sake of compactness.

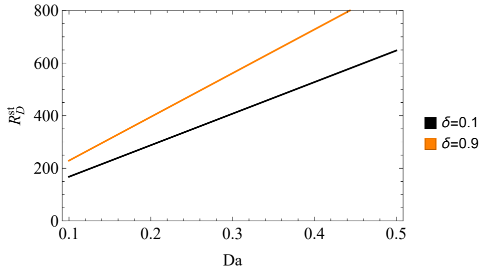
The detailed behaviour of stationary and oscillatory critical Darcy-Rayleigh num-



(a) free-free

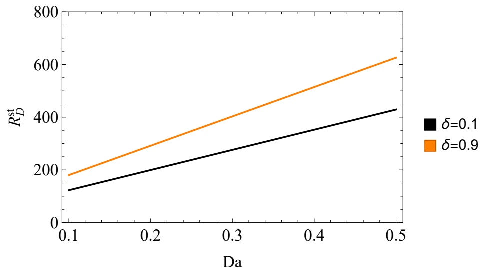


(b) rigid-rigid

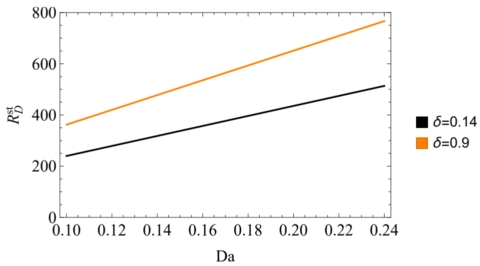


(c) rigid-free

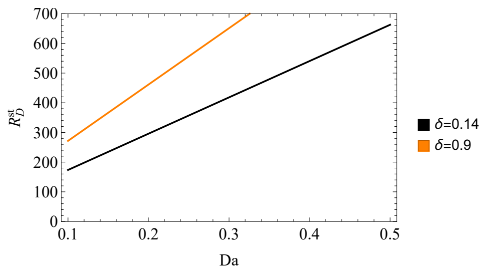
Fig. 4: Plots of $R_D^{\rm st}$ versus Da for *linear* viscosity variations with different values of δ Parameter values: $Da = 0.1$, $\Pr_D = 10$.



(a) free-free



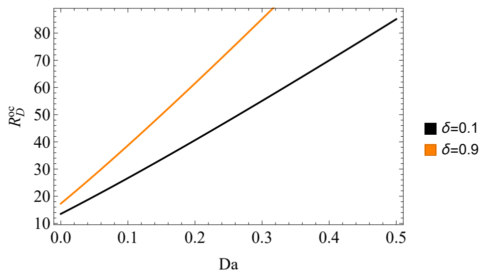
(b) rigid-rigid



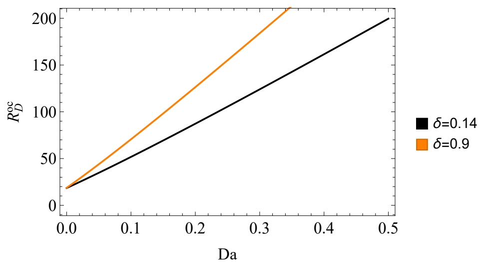
(c) rigid-free

Fig. 5: Plots of $R_D^{\rm st}$ versus Da for *exponential* viscosity variations with different values of δ and $\Pr_D = 10$.

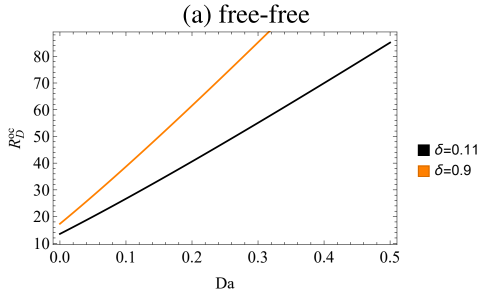
bers with respect to the square of wave number for exponential viscosity variation is analyzed in $R_D^{\rm st}$ *vs.* a^2 and $R_D^{\rm oc}$ *vs.* a^2 planes through Figs. 2-3 for different values of δ and for all cases of boundary conditions. In each of these cases, we found



(a) free-free

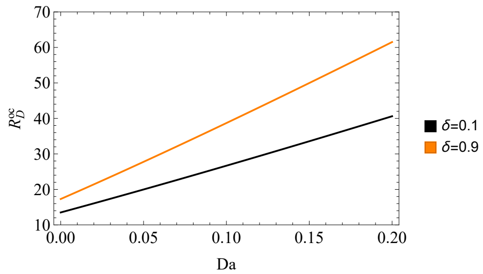


(b) rigid-rigid

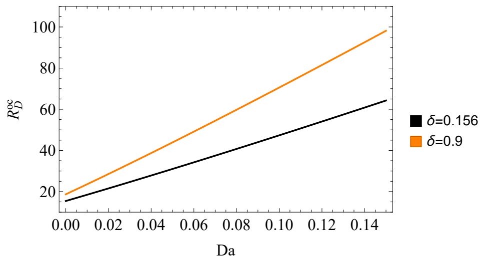


(c) rigid-free

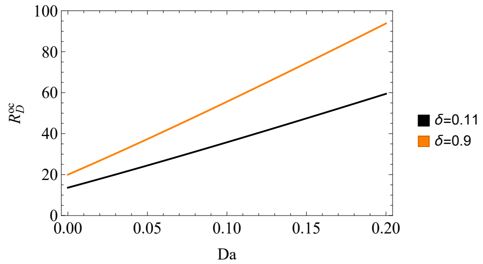
Fig. 6: Plots of R_D^{oc} versus Da for *linear* viscosity variations with different values of δ Parameter values: $\text{Pr}_D = 10$, $\lambda_1 = 0.7$, $Q = 1/7$.



(a) free-free



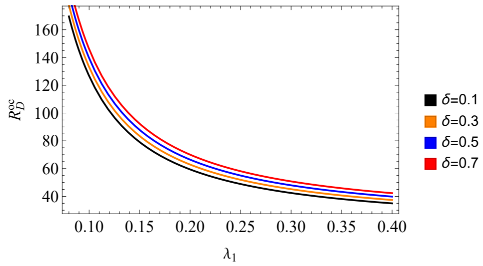
(b) rigid-rigid



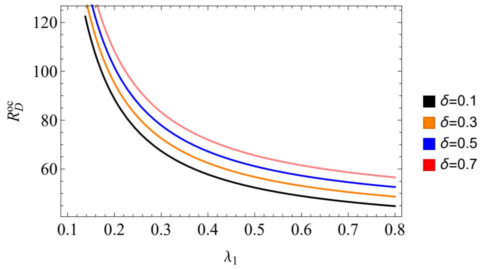
(c) rigid-free

Fig. 7: Plots of R_D^{oc} versus Da for *exponential* viscosity variations with different values of δ Parameter values: $\text{Pr}_D = 10$, $\lambda_1 = 0.7$, $Q = 1/7$.

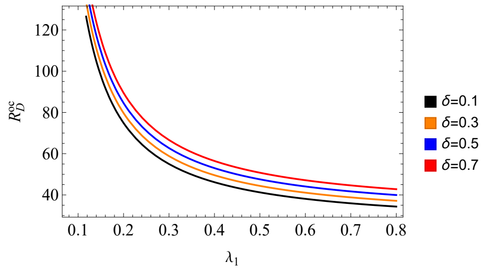
that there is a threshold value for a^2, say a_c^2 at which R_D^{st} *vs.* a^2 and R_D^{oc} *vs.* a^2 attains minimum. For $a^2 < a_c^2$, the critical stationary and oscillatory Darcy-Rayleigh number is a decreasing function of the Darcy-Rayleigh number while for $a^2 > a_c^2$,



(a) free-free

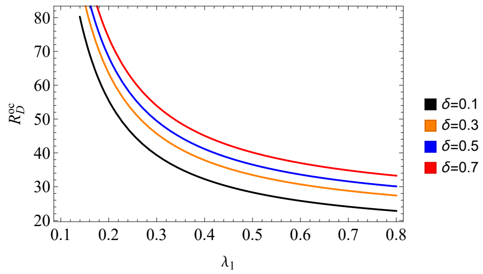


(b) rigid-rigid

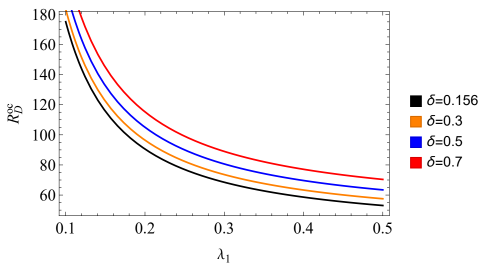


(c) rigid-free

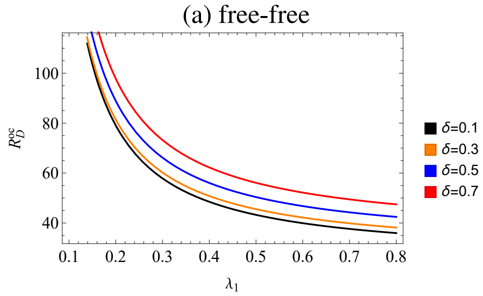
Fig. 8: Plots of $R_D^{\rm oc}$ versus λ_1 for *linear* viscosity variations with different values of δ Parameter values: $Da = 0.1$, $\mathrm{Pr}_D = 10$, $Q = 1/7$.



(a) free-free



(b) rigid-rigid



(c) rigid-free

Fig. 9: Plots of $R_D^{\rm oc}$ versus λ_1 for *exponential* viscosity variations with different values of δ Parameter values: $Da = 0.1$, $\mathrm{Pr}_D = 10$, $Q = 1/7$.

it is increasing. The values of critical $R_D^{\rm st}$ and $R_D^{\rm oc}$ increase with δ which shows that the effect of temperature-dependent viscosity variation (δ) has a stabilizing effect on this generalized system, for all cases of boundary conditions for exponential viscosity variation.

The effect of the modified Darcy number on stationary convection is shown in Figs. 4-5 for *linear* and *exponential* viscosity variation respectively demonstrates the stabilizing nature of modified Darcy number for all cases of boundary conditions. Similarly, the effect of the modified Darcy number on oscillatory convection is shown in Figs. 6-7 for *linear* and *exponential* viscosity variation respectively which demonstrates the stabilizing nature of modified Darcy number for all cases of boundary conditions. Further, it is observed that with increasing values of δ; R_D^{st} and R_D^{oc} increases which shows that the effect of variable viscosity is to stabilize the system for all cases of boundary conditions.

The effect of variable viscosity δ on R_D^{oc} *vs.* λ_1 presented in Figs. 8-9 reveal that R_D^{oc} increases with δ, which shows that the effect of δ is to stabilize the system for all cases of boundary conditions for *linear* and *exponential* viscosity variation. It is also clear that R_D^{oc} decreases with an increase in λ_1 which indicates that the destabilizing effect of λ_1 on the onset of convection, whereas R_D^{oc} increases with the increasing λ_2, which means that the strain retardation time delays the onset of oscillatory convection in a viscoelastic fluid layer.

5 Conclusions

In the present analysis, we have investigated the effects of *linear* and *exponential* temperature-dependent viscosity variations on the onset of thermal convection in Oldroydian viscoelastic fluid in a porous medium. The main conclusions are:

1. If $\sigma_r \geq 0$ and $R_D \leq \dfrac{\pi^2}{\mathrm{Pr}_D \, \lambda_1 \, (1-Q)}$, then we have $\sigma_i = 0$, a sufficient condition for the validity of the principle of exchange of stabilities (*PES*).

2. As usual, the viscoelastic parameters do not affect the onset of stationary convection. Further, the wave number for the case of stationary convection is independent of temperature-dependent viscosity.

3. In all cases of boundary conditions, the temperature-dependent viscosity has a stabilizing effect on the onset of oscillatory convection for the positive values of δ. It is also evident from the obtained results that the *exponential* variation of temperature-dependent viscosity is more stabilizing than the *linear* variation of viscosity.

4. The stress relaxation time (λ_1) destabilize the system whereas the strain retardation time (λ_2) has a stabilizing influence. This may bebecauset the relaxation time reduces the shear rate (i.e., increases the elasticity of a viscoelastic fluid) thus causing instability.

5. The modified Darcy number (Da) has a stabilizing effect on both stationary and oscillatory convection, for *linear* and *exponential* viscosity variations for all cases of boundary conditions. This is because increasing Da amounts to an increase in the viscous effect which in turn retards the fluid flow. Therefore, higher heating is required for the onset of convection with increasing Da. Further, high values of porosity (ϵ), have a destabilizing effect on the onset of convection.

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